Direct numerical simulations of capillary wave turbulence. Supplementary Material: estimation of numerical dispersion and dissipation.

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Summary. The main paper discusses the influence of discretization errors that naturally appear when solving the 3D Navier Stokes equations with current available numerical methods for two phase flows. These errors are linked to the finite size of the grid size and their importance decreases as the mesh is refined. The first source of error is introduced because waves with a wavelength of the order of the grid size are not properly resolved and the surface tension method becomes unaccurate. These inaccuracies are typically translated into errors on the wave propagation frequency and wave damping. The second source of error is directly related to the spatial discretization of the Navier-Stokes equations, which introduce a certain numerical dissipation in the solution that has to be evaluated. In this section we quantify these errors by performing 2D simulations of a propagating single capillary wave between air and water. This problem has an analytical solution for the frequency, given by the linear dispersion relation $f_{rd} = \sqrt{(\gamma/\rho)k^3}/(2\pi)$, and also for the damping on the oscillation, the wave height decays exponentially with a decay rate $\Gamma_{th} = 2\nu k^2$ [1]. The comparison against the theoretical solution allows us to estimate the relative error introduced in the simulation on the wave frequency and wave damping rate as a function of the resolution of the wave. The interface resolution goes from 512 to 2 points per wavelength. In order to compare the 2D and 3D results, we introduce a non-dimension resolution $\delta x/\lambda = k/k_{max}$, where δx is the grid size, λ the wavelength, $k = 2\pi/\lambda$ the wavenumber and $k_{max} = 2\pi/\delta x$ the highest wavenumber resolved. This number corresponds to the inverse of the number of grid points per wavelength. Thus, the range tested is $k/k_{max} \in [0.002 : 0.5]$.

Numerical results show that the numerical dispersion is responsible for a 1% shift from the linear dispersion relation of capillary waves $\omega^2 = (\gamma/\rho)k^3$ when less than 8 grid points are present per wavelength. This threshold will be shown to be in agreement with the shift observed in 3D simulations.

The linear dissipation is estimated by measuring the dissipation rate of wave energy (e.g. amplitude). The wave amplitude evolution fits well an exponentially decay $\eta = \eta_0 e^{-\Gamma t}$. When the resolution is high enough, the theoretical linear viscous decay $\Gamma_{th} = 2\nu k^2$ is recovered. When the wave is poorly resolved, an exponential decay still applies but it is required to empirically fit the numerical dissipation function obtained from simulations as: $\Gamma^{num} = 1/\tau_d^{num} = \mathcal{A}(k/k_{max})\Gamma_{th}$, with $\mathcal{A}$ being an adjusted non-dimensional numerical factor ($\mathcal{A} \approx 100$). This dissipation function describes the total dissipation, i.e. the sum of the numerical dissipation and physical dissipation. As already mentioned the characterization of this dissipation function is very important to interpret the high frequency cut-off of the inertial range and the broadening of the space time wave spectrum at high frequencies obtained from the 3D simulations.

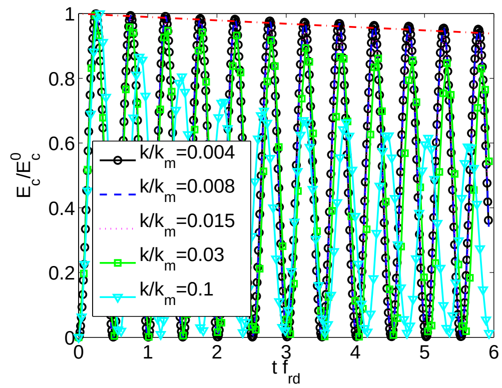

FIG. 1. Color online. Evolution of the kinetic energy (normalized by its maximal value) of a 2D capillary wave as a function of time tf_{rd} for $\delta x/\lambda$ = 0.004,0.008,0.015,0.03,0.1 (from top to bottom). The envelope decays exponentially for all resolutions. At high resolutions ($\delta x/\lambda < 0.015$), this decay follows $E_t \sim e^{-2t\Gamma_{th}}$, while at low resolution ($\delta x/\lambda > 0.015$), a faster decay is visible. A shift on the observed propagation frequency at low resolution is also observed.

2D Decay of a capillary wave. A 2D capillary wave initialized with an amplitude $\eta(x) = \eta_0 \cos(kx)$ freely evolves in a box of size $L = 1$ m. Periodic boundary conditions are applied along the wave propagation direction whereas slip (symmetric) boundary conditions are imposed at the top and bottom of the domain. Four set of wave numbers are tested, $k = N_\lambda\, 2\pi$, with $N_\lambda = 1, 2, 3, 4$ the number of wave period in the box. The wave amplitude is set to $\eta_0 = 0.01L$, so that the wave amplitude remains small and the response is linear. The physical parameters are the same as in the 3D simulation (surface tension γ, density and viscosity). Various resolutions are

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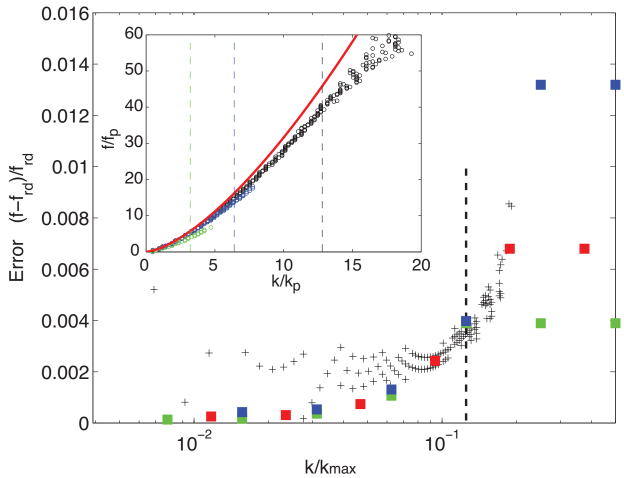

FIG. 2. Color online. Main: frequency error $(f - f_{rd})/f_{rd}$ as a function of the wave resolution k/k_{max}. +: 3D capillary wave turbulence simulation, local maxima of the space time spectrum. Square: 2D simulation of one wave, ■: $N_\lambda = 4$, ■: $N_\lambda = 3$, ■: $N_\lambda = 2$. Vertical dashed lines show the threshold $(k/k_p)/k_{max} < 1/8$ Inset: 3D dispersion relation for three interface resolutions $\delta x = 0.015$ m: (○), $\delta x = 0.008$ m: (○) and $\delta x = 0.004$ m (○); − linear theoretical dispersion relation. Vertical dashed lines show the threshold $(k/k_p)/k_{max} < 1/8$ in each case.

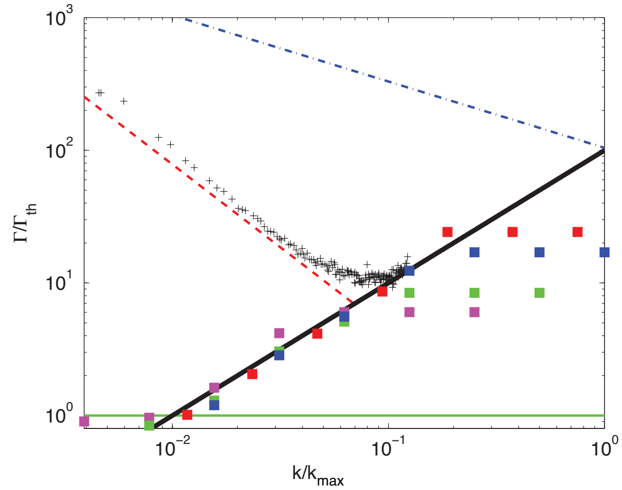

FIG. 3. Dissipation rate Γ/Γ_{th} in the 2D simulations as a function of resolution k/k_{max}. ■: $N_\lambda = 4$, ■: $N_\lambda = 3$, ■: $N_\lambda = 2$, ■: $N_\lambda = 1$. Horizontal solid line: theoretical decay rate $\Gamma/\Gamma_{th} = 1$. Also shown, the width $\Delta\omega/\Gamma_{th}$ (○) of $S_\eta(k, f)$ in the 3D simulation (fig. 4, main paper); (−−): $\sim k^{3/4}$; (− · −): ω/Γ_{th}, $\omega = \sqrt{(\gamma/\rho)k^3}$ linear dispersion relation.

tested from $\delta x/\lambda \in [0.001 : 0.5]$. In both fluids adaptive refinement is used with the same conditions as in the 3D simulations.

Figure 1 shows the evolution of the 2D capillary wave kinetic energy ($N_\lambda = 2$) for various grid sizes as a function of time tf_{rd}. The kinetic energy oscillates at half the wave frequency and its amplitude decreases exponentially in time. At high resolution ($\delta x/\lambda < 0.015$), the energy decay is well described by the classic viscous damping, and the frequency propagation fits well the linear dispersion relation. The wave is correctly resolved both in amplitude and frequency. At low resolution ($\delta x/\lambda > 0.015$), a shift in frequency occurs that become more pronounced as the resolution is reduced. The wave energy also decreases faster than expected due to numerical dissipation. The decay rate can be estimated fitting the energy decay to $E_c(t) = E_0 e^{-\Gamma t}$, where $\Gamma(k, \delta x)$ depends on both the wave number and the resolution. These observations hold for all N_λ and the various kinematic viscosity values tested.

Numerical dispersion

We now quantify in Fig. 2 the error in the frequency as a function of resolution measured as k/k_{max}. The relative error is computed according to $(f - f_{rd})/f_{rd}$, where f_{rd} represents the theoretical frequency value. In addition to the 2D test case described above (squares) we also include the error found from the full 3D simulation of capillary wave turbulence contained in the main manuscript (crosses). For 3D simulations, waves at various frequencies spread all over the space time spectrum and the numerical dispersion relation $f(k)$ corresponds to the local maximum of $S_\eta(k, f)$ (see + on Figure 2 in the main paper). In both cases the error on the frequency resolution is smaller than 1% when $k/k_{max} < 1/8$, that is, a significant error is introduced for waves with less than 8 points per wavelength, whereas accurate numerical results are obtained for larger wavelengths. The inset of Figure 2 shows the $f(k)$ function obtained from 3D simulations as a function of the interface resolution: 64×64 ($\delta x = 0.015$ m), 128×128 $\delta x = 0.008$ m et 256×256 $\delta x = 0.004$ m. In all cases the threshold at which we find a significant shift from the linear dispersion scales with $k/k_{max} \approx 1/8$. Thus the shift observed at high frequencies in the 3D computations have been identified to be related to interface discretization errors. This error remains small for waves resolved with more than 10 grid points.

Numerical dissipation

We have seen in Fig. 1 that when a wave is not well resolved, the dissipation rate is larger than expected. The damping rate $\Gamma(k, k_{max})$ is systematically calculated as a function of k/k_{max} in the 2D simulations (Figure 3). For $k/k_{max} < 0.01$, the wave damping is correctly solved and is found in agreement with the classical viscous damping. When the wave is not well resolved, an extra damping is observed and an empirical dissipation function can be estimated by fitting the numerical results (black solid line in Figure 3). This numerical dissipation function depends on the physical dissipation and also on the grid size applied to the interface and is expressed as:

$$\Gamma^{num} = 1/\tau_d^{num} = \mathcal{A}\frac{k}{k_{max}}\Gamma_{th}, \quad (1)$$

with $\mathcal{A} = 100$ a fitted non-dimensional constant. We have checked that this function is valid irrespective of the physical viscosity value. The numerical dissipation function is also compared with the broadening $\Delta\omega = 1/\tau_{nl}$ of the space time spectrum (Figure 4 in the main paper). When the non linear time scale $(1/\tau_{nl})/\Gamma_{th}$ and the numerical dissipation time Γ^{num}/Γ_{th} become of the same order ($k/k_{max} \approx 0.07$) the energy cascade can no longer be observed as the dissipation is responsible for the broadening of the spectrum ($k/k_{max} > 0.06$). This corresponds to the range of scale where the empirrical numerical dissipation is used to calculate the mean energy flux ($k/k_p > 6$) in the main paper. However the numerical dissipation does not affect the capillary wave turbulence cascade for lower wave number as shown by the observed time scale separation $1/\tau_{nl} \gg 1/\tau_d^{num}$. Finally, note that we only need to introduce the numerical dissipation on the discussion of the results because the resolution of the simulation is not high enough and the the numerical dissipation controls the extent of the inertial range.

[1] H. Lamb, *Hydrodynamics* (Dover, 1932).