Supplemental Material
"Saturation of the inverse cascade in surface gravity wave turbulence"

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In this Supplemental Material, we present movies (§1) and additional data analyses related to the observation of the inverse cascade in gravity wave turbulence on the surface of a fluid: Wave homogeneity (§2), wave height distribution probability (§3), and timescale separation (§4). Notations as in the aforementioned paper.

1. MOVIES

- BasinMEDcutLow.mpeg (35s - 24Mo): Large-scale wave basin seen from the shore showing the wave makers, the wall at the opposite end, the probe array, and the control room. Weak random forcing conditions.
- WeakForcingIMG_7799.mp4 (15s - 17Mo): Wave field seen from the shore. Weak random forcing conditions: Wave steepness $\epsilon = 0.063$ ($\sigma_\eta^{sat} = 0.59$ cm) and narrow spectral bandwidth (1.8 ± 0.2 Hz).
- HigherForcingIMG_7800.mp4 (15s - 16Mo): Wave field seen from the shore. Higher random forcing conditions: Wave steepness $\epsilon = 0.98$ ($\sigma_\eta^{sat} = 1.84$ cm) and narrow spectral bandwidth (1.8 ± 0.2 Hz).

2. WAVE HOMOGENEITY

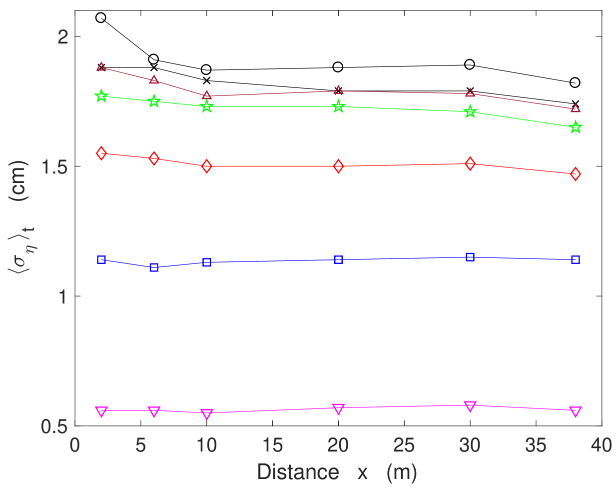

FIG. 1: Time-averaged standard deviations of wave height $\sigma_\eta(x)$ recorded at different distances x from the wave makers for different forcing amplitudes during 60 min. Good homogeneity for weak enough forcing. Same symbols as in the main paper.

3. WAVE PROBABILITY DISTRIBUTION

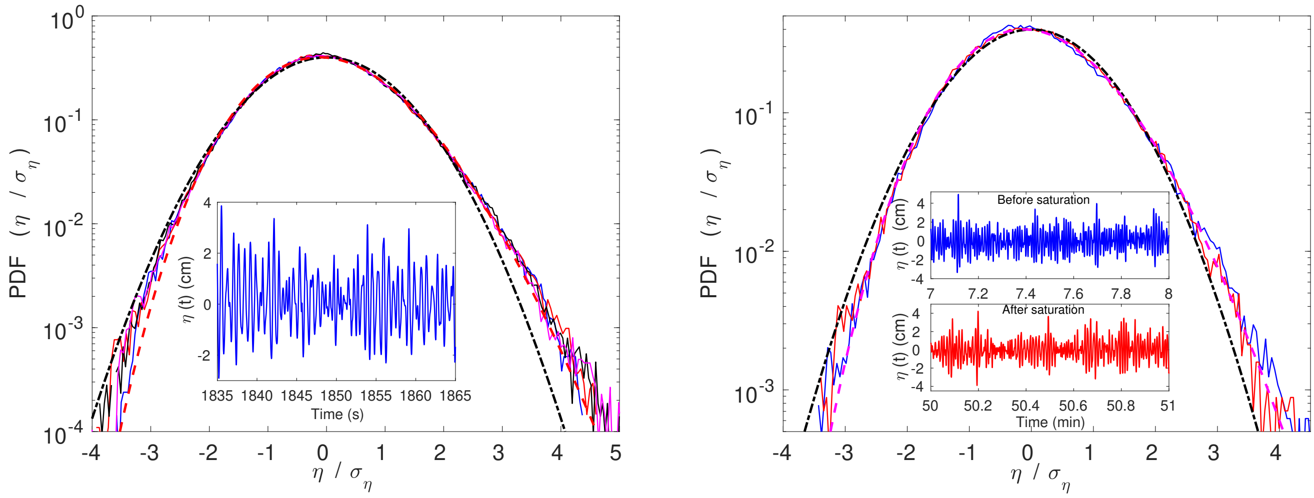

FIG. 2: **(Left)** Probability distribution functions (PDF) of normalized wave height $\eta(t)/\sigma_\eta$ recorded in the middle of the basin ($x = 20$ m) during 60 min and averaged on different probes [n°5 (blue), 10 (red), 11 (black), and 12 (magenta)]. $\sigma_\eta \equiv \sqrt{\langle \eta(t)^2 \rangle_t} = 1.14$ cm. Black dash-dotted line displays a Gaussian of zero mean and unit standard deviation. Red dashed line shows a Tayfun distribution for a wave steepness of 0.08. A weak asymmetry $S = \langle \eta^3 \rangle / \langle \eta^2 \rangle^{3/2} = 0.23$ and a weak Kurtosis $F = \langle \eta^4 \rangle / \langle \eta^2 \rangle^2 = 3.3$ are observed as expected for a weakly nonlinear wave field (Normal distribution would lead to $S = 0$ and $F = 3$). Inset displays a part of the corresponding signal $\eta(t)$ for probe n°12. **(Right)** PDF of $\eta(t)/\sigma_\eta$ computed before ($t \leq 16$ min - blue) and after ($35 < t \leq 51$ min - red) the saturation of the inverse cascade, and parts of the corresponding temporal signals $\eta(t)$ (insets). $x = 20$ m (probe n°12). $\sigma_\eta = 1.14$ cm

4. WAVE TURBULENCE TIMESCALE SEPARATION

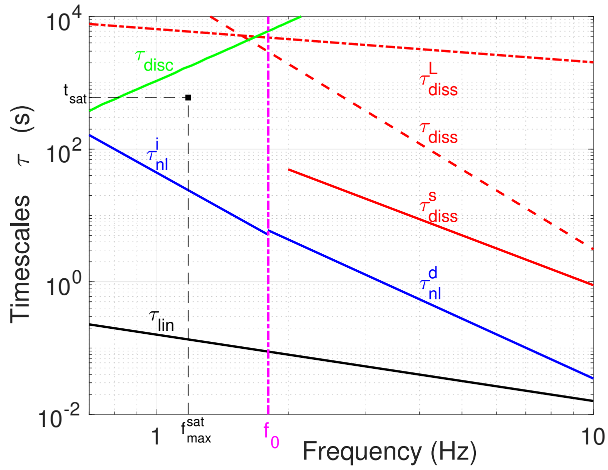

FIG. 3: Wave turbulence timescale separation. (Black solid line) Linear propagation timescale $\tau_{\rm lin} = 1/\omega$. (Blue solid lines) Nonlinear interaction timescales: $\tau^i_{nl} = c^i g^{1/6} Q^{-2/3} k(\omega)^{-11/6}$ (inverse cascade) and $\tau^d_{nl} = c^d g^{1/2} P^{-2/3} k(\omega)^{-3/2}$ (direct cascade) [4], using the values of $Q(\epsilon)$ and $P(\epsilon)$ inferred experimentally [fixed wave steepness $\epsilon = 0.1$ ($\sigma_\eta^{sat} = 1.92$ cm)], the dimensionless constant value $c^d = 0.03$ found experimentally [28], and assuming $c^i = c^d$. (Green solid line) Discreteness time τ_{disc} computed as the number of eigenmodes found in a frequency band divided by this bandwidth, taking into account both transverse and lateral eigenmodes. No discreteness effect for $\tau_{nl} < 2\tau_{disc}$ (i.e. nonlinear spectral widening > half frequency separation between adjacent eigenmodes). Linear viscous dissipation timescales [36]: (red dashed line) viscous surface $\tau_{diss} = 1/[2\nu k(\omega)^2]$, (red solid line) surface boundary layer with an inextensible film $\tau^s_{diss} = 2\sqrt{2}/[k(\omega)\sqrt{\nu\omega}]$ negligible for $f \lesssim 2$ Hz [31], and (red dotted-dash line) lateral boundary layer $\tau^L_{diss} = 2\sqrt{2} L_x L_y / [3\sqrt{\nu\omega}(L_x + L_y)]$, $L_x = 40$ m, $L_y = 30$ m, whereas bottom boundary layer is negligible. Water kinematic viscosity $\nu = 10^{-6}$ m^2/s. f_0 is the central forcing frequency, and f^{sat}_{max} the frequency of the spectrum maximum at the saturation time t_{sat}. $\omega(k) = \sqrt{gk}$.