On scaling laws in granular flows down a rough plane

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The scaling properties of granular flows down an inclined plane are investigated in a model previously proposed to describe surface flows on a sandpile. Introducing a depth dependant friction, we are able to reproduce the results obtained experimentally by [O. Pouliquen, Phys. Fluids **11**, 542 and 1956 (1999)] on both the fronts velocities and their shapes.

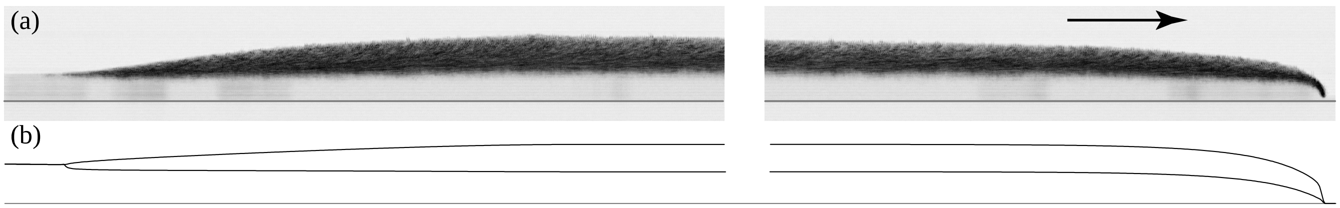

FIG. 1. **(a) Experiment:** side view of a dry granular flow down an inclined plane covered by velvet cloth. Right: a sharp front propagates at a constant velocity (from left to right on the figure), with a steady shape. Successive images difference, reveal that the grains flow only on part of the total height (black), the rest (light gray) being static. Left: when the injection is suddenly stopped, a stopping front propagates downward (from left to right on the figure), at a larger velocity for the case presented here. A uniform static layer of thickness Z^*_{stop} finally remains on the plane. The vertical scale is 10 times the horizontal one. **(b) Model:** the model is integrated numerically for the same conditions. As in the experiment, starting and stopping fronts propagate downward at a constant velocity.

In two recent articles [1,2], Pouliquen has remarkably shown in an experiment that granular avalanches down a rough inclined plane exhibit a robust scaling law, valid for various systems of beads and various plane roughness. This scaling shows that the only characteristic length-scale is the thickness $Z^*_{stop}(\varphi)$ remaining when the flow stops [3,4], (see fig. 1). Despite its simplicity, this scaling has not yet received any theoretical explaination.

On the other hand, several models have been recently proposed to describe granular flows at the surface of a sandpile [5,6]. The DAD model is derived from macroscopic conservation laws integrated along the vertical direction (Saint-Venant equations) [7] and is based on the rigorous derivation of the effective force acting on one rolling grain [8]. The aim of this letter is to compare the rheology obtained with our model, *a priori* not constructed for the case of an avalanche on a fixed bottom, to Pouliquen scaling law.

We will give here a quick overview of our model, the detailled derivation and the effect of each term being postnoted to a forthcoming article. The evolution of the free surface ζ (fig. 2) is given by the conservation of matter:

$$D_t \zeta = 0 \qquad (1)$$

where D_t denotes the material derivative [9]. The evolution of the momentum $\vec{q} = \frac{1}{2} H^2 \vec{\Gamma}$ (fig. 2) is given by the dynamical equation, the forces being pressure, gravity and the effective friction $\mu_Z(H)$ acting on the bottom of the flowing layer (fig. 3) :

$$D_t \vec{q} = -gH \left(\vec{\nabla} H + \cos\theta \, \vec{\nabla} Z + \mu_Z \cos\theta \vec{s} \right) \qquad (2)$$

where θ is defined by $\tan\theta = |\vec{\nabla} Z|$.

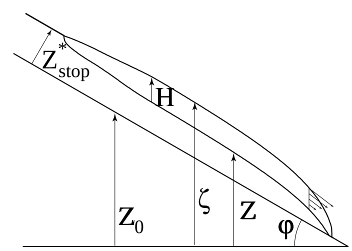

FIG. 2. In the model, the velocity profile inside the flowing granular layer is assumed to be linear, its vertical gradient being $\vec{\Gamma} = \Gamma \vec{s}$ ($\vec{s}$ is the direction of motion). The local state of the sand pile is described by the flowing depth H, the free surface profile ζ and by the position of the static/flowing interface $Z = \zeta - H$. H^*, Z^* et ζ^* are the same quantities but measured from the plane, perpendicularly to it.

The momentum $\vec{q}$ increases either when the flowing height H (the mass) increases or when the velocity gradient $\vec{\Gamma}$ (and thus the mean velocity) increases: $D_t \vec{q} = \frac{1}{2} H^2 D_t \vec{\Gamma} + \vec{\Gamma} H D_t H$. Correspondingly, the right hand side of eq. (2) is splitted into two contributions [9]. The evolution of the velocity gradient inside the flowing layer results from the balance between gravity driving and dissipation by collisions $\mu_\Gamma(\Gamma)$:

$$\frac{H}{2} D_t \vec{\Gamma} = -g \left(\vec{\nabla} H + \cos\theta \vec{\nabla} Z + \mu_\Gamma \cos\theta \vec{t} \right) \qquad (3)$$

The evolution of the interface between the static and flowing layers is governed by the competition between the collisions which tend to mobilise the static part and the trapping of grains in the holes between the underneath ones [8]:

$$\Gamma D_t H = g \cos\theta(\mu_\Gamma - \mu_Z) \quad (4)$$

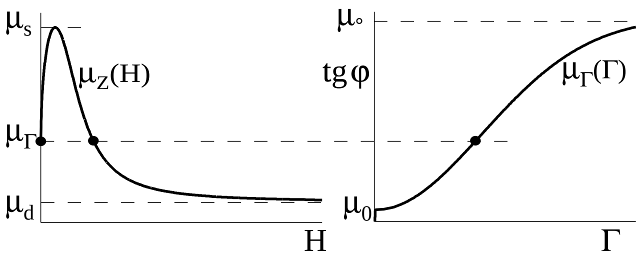

FIG. 3. **a)** the flowing layer is globally submitted to an effective friction $\mu_Z(H)$ which characterises the force needed to mobilise particles from the static part of the sandpile. **b)** the velocity gradient $\vec{\Gamma}$ at an angle φ is determined by the balance between gravity and dissipation by collisions $\mu_\Gamma(\Gamma)$.

$\mu_Z(H)$ is constructed to capture the hysteretical character of the transition between the static and flowing states [4,8], and the decrease of effective friction with velocity. To take into account the fact that the rough bottom spreads its influence inside the grain layer up to several grains diameters [3,4], *the friction coefficients μ_d and μ_s are assumed to depend exponentially on the position* Z^* of the static/flowing interface with respect to the solid bottom (fig. 4 a).

$\mu_\Gamma(\Gamma)$ directly derives from the analysis of the motion of one grain [10,8]: when gravity is balanced by dissipation the grain reaches a constant velocity Γ which depends simply on the plane angle φ. This equilibrium gives $\mu_\Gamma(\Gamma) = \tan\varphi$ (fig. 3 b).

To adjust our model parameters to the Pouliquen's experiments, we first computed numerically the height $Z^*_{stop}(\varphi)$ remaining on the plane when the flow stops (fig. 4), as shown in fig. 1 b (left). The surface profile let behind it by the stopping front is not strictly parallel to the plane but tends towards a constant height Z^*_{stop} as $1/x^*$. The thickness $Z^*_{stop}(\varphi)$ is the same whatever the initial condition is, but the $1/x^*$ decrease is more pronounced when inertia is initially large i.e. at large flowing height H (fig. 1).

In first approximation $Z^*_{stop}(\varphi)$ gives the dynamical friction coefficient $\mu_d(Z^*)$ [1,2,4]: when the flowing height is decreased, Z^*_{stop} is the first layer with enough friction to stop. It can be observed however that $Z^*_{stop}(\varphi)$ is slightly larger than the dynamical friction coefficient $\mu_d(Z^*)$ (fig. 4). This is due to the hysteresis, which induce an increase of friction at small flowing height (fig. 3 a).

We also measured, as done by Pouliquen, the angle φ_{stop} below which a flow of given height ζ^* stops. As in Pouliquen's experiment, the stopping height $Z^*_{stop}(\varphi)$ and $\varphi_{stop}(\zeta^*)$ nearly collapse (fig. 4).

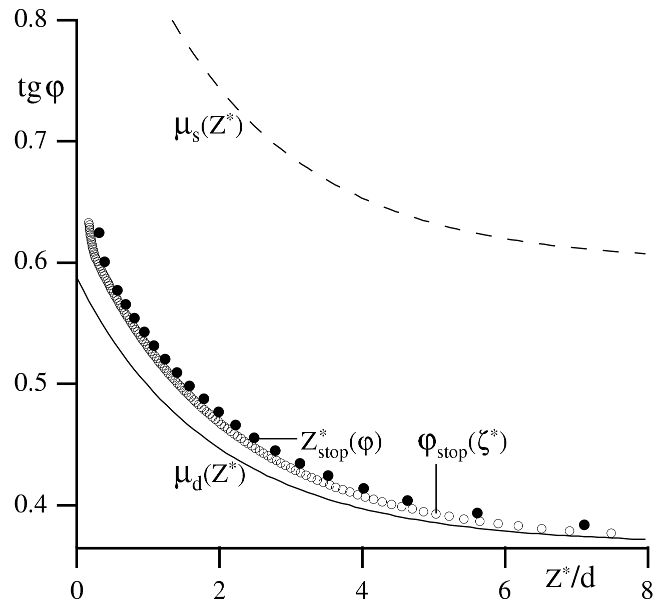

FIG. 4. After an avalanche, a static layer of height $Z^*_{stop}(\varphi)$ remains on the plane (black dots). It is nearly equal, but slightly above the minimum angle φ_{stop} below which any flow stops (white circles). The friction coefficients $\mu_s(Z^*)$ and $\mu_d(Z^*)$ were adjusted to recover the measurement of Pouliquen (system 1).

We can now compare the front shapes obtained for different flow rates q at the same angle φ. When rescaled by the total height far from the front ζ^*_∞, the front profiles nearly collapse on the same curve (fig. 5). Just at the front however, in the model the slope slightly increases with q. Globally we observed that the slope at the front decreases with increasing plane angle φ, when computed with respect to the plane (quantities noted with a star). On the other hand, the slope with respect to gravity (fig. 2) remains almost constant, as observed by Pouliquen [2]. Far from the front (at ∞), the free surface profile and the interface between static and flowing layers become parallel to the plane. The balance between mobilisation and trapping determines the position Z_∞ of the static/flowing interface $\tan\varphi = \mu_Z(H)$. The internal equilibrium between gravity and dissipation by the collisions fixes the velocity gradient Γ ($\tan\varphi = \mu_\Gamma(\Gamma)$). The model thus basically predicts that the grains flow only on a part of the total height (fig. 1 b) with a static thickness Z^*_∞. This situation was not imagined by the previous authors who considered a layer flowing down to the solid rough bottom. An experiment with lateral glass plates (fig. 1 a) show that it is indeed the case.

This static height is determined by the equilibrium between the gravity effects and the friction. This friction increases close to the rough bottom. As the gravity effect increases with the plane angle, this static height thus decreases (and eventually vanishes above the slope $\mu_d(0)$) (fig. 4). On the other hand, except for the small hysteresis, this friction is almost independant of the flowing height (fig. 3). So the static height is determined by the plane angle, but only slightly by the flowing height above it. This system is thus similar to the system imagined previously, with a flow down to a fixed bottom, expect that this rough bottom is now at a height Z^* depending on the plane angle.

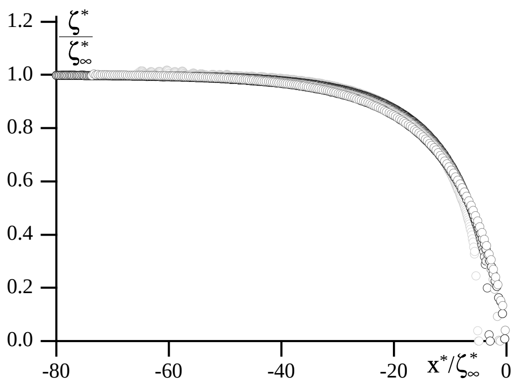

FIG. 5. Front profiles $\zeta^*(x^*)$ rescaled by the total height far from the front ζ^*_∞ for $\varphi = 24°$. For the five flow rates q shown (1/4, 1, 4, 16 and 64 in units of $g^{\frac{1}{2}} d^{\frac{3}{2}}$), the fronts profiles nearly collapse on the same curve.

The front velocity u^* is fully determined by the global conservation of matter, $\zeta^*_\infty u^* = q = \Gamma H^2/2$ and thus does not depend on the front shape. As suggested by Pouliquen results, u^* can be rescaled by the gravity g and the avalanche height ζ^*_∞ to form a Froude number:

$$\frac{u^*}{\sqrt{g\zeta^*_\infty}} = \frac{\Gamma(\zeta^*_\infty - Z^*_\infty)^2}{2\cos^2\varphi\sqrt{g}\zeta^{*\frac{3}{2}}_\infty} \tag{5}$$

which plotted for different angles φ as a function and $\zeta^*_\infty / Z^*_{stop}$ (fig. 6). It turns out that the curves obtained for different angles nearly collapse on a single curve, as obtained experimentaly (it however exhibits a small curvature). The mean slope of the curve is around 0.12 wheras it was 0.136 in the experiment, meaning that the model allows to recover the experimental results both *qualitatively* and *quantitatively*.

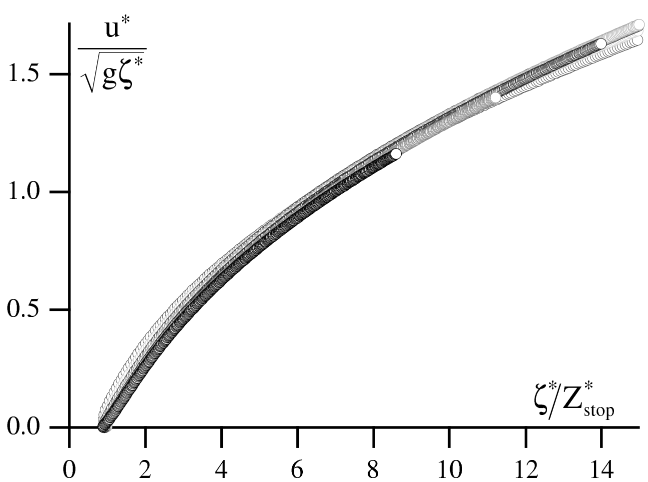

FIG. 6. Dimensionless front velocity $u/\sqrt{g\zeta'}$ as a function of ζ/ζ_{stop} for different inclination angles (every 1° between 22° and 28°).

The DAD model, derived for thick piles, can thus be adaptated to simulate flows on a rough bottom by introducing a depth dependent friction. For instance we recover the scaling of the front shape and velocity measured in experiments. However, this model (and an experiment) reveal that the flow does not occur down to the bottom plane, and that there exists a layer of static grains. This suggests that the effective rheology derived assuming that the flow occurs on the whole height should be modified by taking this static layer into account.

Acknowledgments

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[9] Due to the linear velocity profile, the conservation of matter can be expressed by $D_t\zeta = \partial_t\zeta + \vec{\nabla}\cdot\vec{q} = 0$, where $\vec{q} = \frac{1}{2}H^2\vec{\Gamma}$ is the flow rate. Similarly, the material derivative of the flow rate $\vec{q}$ is related to the energy tensor $\tilde{E} = \frac{1}{3}H^3\vec{\Gamma}\otimes\vec{\Gamma}$ by $D_t\vec{q} = \partial_t\vec{q} + \vec{\nabla}\cdot\tilde{E}$. By definition, the velocity is null at the static/flowing interface $Z = \zeta - H$ which is not advected. Thus, the material derivative of the flowing height is $D_tH = \partial_tH + \vec{\nabla}\cdot\vec{q}$. Finally, the convection term of eq. (3) derives from the two previous ones (eqs. 2 and 4) and reads: $D_t\vec{\Gamma} = \partial_t D_t\vec{\Gamma} + H\left(\frac{2}{3}\vec{\nabla}\cdot(\vec{\Gamma}\otimes\vec{\Gamma}) - (\vec{\nabla}\cdot\vec{\Gamma})\vec{\Gamma}\right)$.

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