4.7 Mathematical Example of the Limit Re $\to \infty$

Since the discussion above deals with some of the most important fundamentals of boundary–layer theory, let us elucidate these ideas further with a very simple mathematical example given by L. Prandtl[1].

We consider the oscillation of a point mass with damping, given by the differential equation

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = 0 \,. \tag{4.62}$$

Here m is the oscillating mass, k the damping constant, c the spring constant, x the distance of the mass from the position of rest, and t the time.

The initial conditions are given as

$$t = 0: \quad x = 0 \,. \tag{4.63}$$

In analogy to the Navier–Stokes equations with very small kinematic viscosity ν, we consider the limit of very small mass m, since then in Eq. (4.62) too the highest order term becomes very small.

The complete solution of (4.62) with initial condition (4.63) reads

$$x = A[\mathrm{e}^{-(ct/k)} - \mathrm{e}^{-(kt/m)}] \quad m \to 0 \,, \tag{4.64}$$

where A is a free constant which could be fixed by a second initial condition.

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[1] L. Prandtl, Anschauliche und nützliche Mathematik, lectures Winter semester 1931/32, Göttingen.

If we set $m = 0$ in Eq. (4.62) we obtain the simplified differential equation

$$k\frac{dx}{dt} + cx = 0\,, \tag{4.65}$$

which is now a first order differential equation with solution

$$x_o(t) = A\mathrm{e}^{-(ct/k)}\,. \tag{4.66}$$

By choosing the initially arbitrary constant A suitably, this solution agrees with the first term of the complete solution. However it cannot satisfy the initial condition (4.63). Therefore it is a solution for large times ("outer" solution).

A simplified differential equation can also be derived from Eq. (4.62) for the solution for small times ("inner" solution). To this end a new "inner" variable is introduced by "stretching" the time coordinate t:

$$t^* = \frac{t}{m}\,. \tag{4.67}$$

Using this, Eq. (4.62) reads

$$\frac{d^2x}{dt^{*2}} + k\frac{dx}{dt^*} + m\,c\,x = 0\,. \tag{4.68}$$

For $m = 0$ we obtain the differential equation for the "inner solution":

$$\frac{d^2x}{dt^{*2}} + k\frac{dx}{dt^{*2}} = 0\,. \tag{4.69}$$

with the solution

$$x_i(t^*) = A_1\mathrm{e}^{-kt^*} + A_2\,. \tag{4.70}$$

In spite of the simplification, this equation remains second order, and it can satisfy the initial condition (4.63). We then have

$$A_1 = -A_2\,. \tag{4.71}$$

Determining the constant A_2 is carried out by matching up the "inner" solution and the "outer" solution corresponding to Eq. (4.66). The solutions in Eq. (4.66) and (4.70) must be equal in an overlap region, i.e. for intermediate times. It must hold that:

$$\lim_{t^* \to \infty} x_\mathrm{i}(t^*) = \lim_{t \to 0} x_\mathrm{o}(t) \,. \tag{4.72}$$

In words the conforming condition says: the "outer" limit of the inner solution is equal to the "inner" limit of the outer solution. From Eq. (4.72) it immediately follows that

$$A_2 = A \tag{4.73}$$

and thus for the inner solution

$$x_\mathrm{i}(t^*) = A(1 - \mathrm{e}^{-kt^*}) \,. \tag{4.74}$$

This solution can also be obtained from the complete solution in Eq. (4.64) if the first term is expanded for small t and only the first term of the expansion is taken into account, i.e. setting

$$\lim_{t \to 0} \mathrm{e}^{-(ct/k)} = 1 \,. \tag{4.75}$$

The two solutions, the outer solution from Eq. (4.66) and the inner solution from Eq. (4.74) represent the entire solution if each is applied in its region of validity.

At a given t, Eq. (4.64) for $m \to 0$ passes over to the outer solution. We obtain the entire solution valid for the whole t region (the composite solution) from the partial solutions by adding both solutions. Here the part in common to both solutions may only be taken into account once, i.e. it must be subtracted:

$$x(t) = x_{\mathrm{o}}(t) + x_{\mathrm{i}}(t^*) - \lim_{t^* \to \infty} x_{\mathrm{i}}(t^*) = x_{\mathrm{o}}(t) + x_{\mathrm{i}}(t^*) - \lim_{t \to 0} x_{\mathrm{o}}(t) \; . \tag{4.76}$$

In this manner, the composite solution in Eq. (4.64) follows from the two separate solutions.

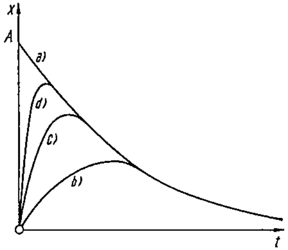

Fig. 4.2. Solution of the oscillator equation (4.62) for $m \to 0$.
(*a*) Solution of the simplified differential equation (4.65), $m = 0$;
(*b*), (*c*), (*d*) Solutions of the complete differential equation (4.62) for different values of m. For small values of m, curve (*d*) depicts a solution with "boundary–layer character"

The composite solution in Eq. (4.64) is shown graphically in Fig. 4.2 for $A > 0$. Curve a is the outer solution. Curves b, c and d are the composite solution, where the value of m decreases as we go from b to d.

If we compare this solution with the Navier–Stokes differential equations, the complete differential equation (4.62) corresponds to the Navier–Stokes differential equations of a viscous fluid; the simplified differential equation (4.65) to the outer solution of the Euler differential equation of an inviscid fluid; and the simplified differential equation (4.69) to the inner solution of the boundary–layer equations which have still to be derived. The initial condition (4.63) corresponds to the no–slip condition of the viscous fluid, which can be satisfied by the solutions of the Navier–Stokes equations but not by the solutions of the Euler equations. The outer solution corresponds to the inviscid outer flow (potential flow) which does not satisfy the no–slip condition at the wall. The inner solution corresponds to the boundary–layer flow which is determined by the viscosity and which is only valid in a narrow zone attached to the wall (boundary layer or frictional layer). However it is only by including this boundary–layer solution that the no–slip condition at the wall can be satisfied, and thus the entire solution make sense physically.

Therefore this simple example has again confirmed the same mathematical idea we saw in the previous section, namely that taking the limit to very small viscosities (very large Reynolds numbers) in the Navier–Stokes equations cannot be carried out by simply eliminating the friction terms in the differential equation. It is performed by first obtaining the solution and then allowing the Reynolds number to become very large.

Later we will see that it is not necessary to retain the complete Navier–Stokes equations when taking the limit $\mathrm{Re} \to \infty$. For reasons of mathematical simplicity, we will be able to assume a number of terms in these equations, in particular friction terms, are small enough to be neglected. However it is important that not all the friction terms are neglected, since this would reduce the order of the Navier–Stokes equations.