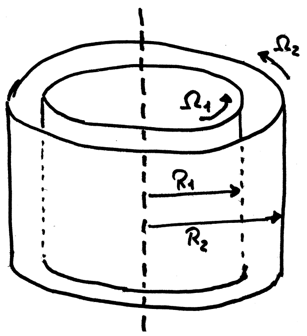
TD1 — Parallel flows in curved geometries

M2 ICFP Physics of Fluids – Adrian Daerr[*]

1 Taylor-Couette flow and viscosimeter



In a Taylor-Couette device[1] a liquid (which we suppose incompressible, of density ρ and dynamic viscosity η) fills the gap between two coaxial cylinders of length L and of radii R_1 and R_2, respectively ($L \gg R_2 - R_1$). We are interested in the stationary flow when both cylinders rotate at uniform angular velocities Ω_1 resp. Ω_2.

1. What are the symmetries in this system ? What are the implications for the flow ? Under what conditions ? Use the incompressibility relation to further narrow down the general shape of the velocity field.
2. Use the equations of motion to calculate the velocity in the gap in the steady state. Does the viscosity play a role ? Why (or why not) ?
3. What is the vorticity of this flow ? What component of the velocity field contributes to it ?
4. Determine the torques exerted by the flow on the inner and outer cylinders. How do they relate ? Check how the torque acts on the inner cylinder when $\Omega_1 = 0$ and $\Omega_2 > 0$, or when $\Omega_1 > 0$ and $\Omega_2 = 0$. Repeat this analysis for the outer cylinder.

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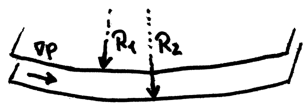
[*]based on notes by Philippe Gondret

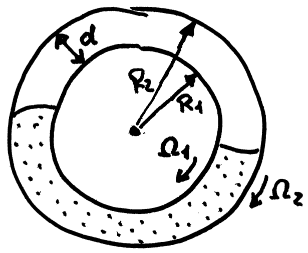
1. Such a device was built by Eugène Ducretet (who also assembled the first wireless telegraph), and then studied experimentally as well as theoretically by Maurice Couette (1858–1943) during his PhD thesis in 1888. It allowed for remarkably precise measurement of the viscosity of water and air. In 1923, G.I. Taylor (1866–1975) showed both theoretically and experimentally that rotating the inner cylinder whilst keeping the outer cylinder fixed leads to an instability of the Taylor-Couette flow through the formation of counter-rotating annular vortices.

2 Bent Poiseuille flow



Let us now consider the flow between two bent plates of radii R_1 resp. R_2 and of transverse dimension $L \gg R_2 - R_1$ (see figure). The flow is now induced by a pressure gradient acting along the azimuthal (orthoradial) direction, and having a magnitude independent of θ. We wish to determine the steady flow, by adequately simplifying, and then solving, the Navier-Stokes equations. What is the expression for the stationary velocity field ?

3 Taylor–Dean flow



Consider finally a cylindrical Couette-Taylor viscosimeter flipped on its side so that the axis is horizontal. This device is now only half-way filled with liquid (see figure). Again we assume that the gap between the coaxial cylinders $d = R_2 - R_1$ is small compared to the length of the latter, and we wish to determine once more the stationary flow when the cylinders are rotated at constant frequencies.

1. How does the system reach the steady state ? How is this problem linked to the two problems studied above ? Far from the free surfaces of the liquid, what can be said about the velocity components u_r and u_θ ? Sketch the velocity field and streamlines in this region. Then draw the velocity field and streamlines near the free surfaces. Using the incompressibility relation, estimate the size of the perturbed regions near the free surfaces.
2. Using the results obtained in the two problems studied earlier, express the velocity field far from the free surfaces without further calculus. Specify the boundary conditions and the additional constraint required to solve the problem.
3. In order to simplify calculations, we now assume the gap and the curvature to be very small, that is $d \ll R_1, R_2$. This allows us to write the flow as a superposition of a *plane* Coutte flow and a *plane* Poiseuille flow. Under these assumptions, show that the velocity field may be written as

$$u_\theta(y) = \Omega_1 R_1[1 - 2(2+\mu)y + 3(1+\mu)y^2]$$

   where $\mu = \Omega_2/\Omega_1$ is the ration of the rotation rates and $y = (r - R_1)/d$ denotes the dimensionless coordinate across the gap.
4. Express the difference in the heights of the free surfaces Δh as a function of the control parameters.
5. Sketch the velocity field for various rotation ratios, e. g. $\mu = 1$, 0 and -1.