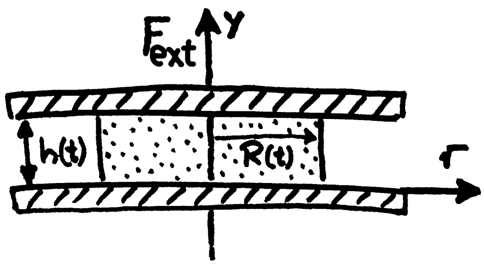
TD2 — Quasi-parallel flow and lubrication

M2 ICFP Physics of Fluids – Adrian Daerr*

1 Hydrodynamics of adhesion

Let us consider a drop of viscous liquid confined between to parallel plates, which are at a distance h much smaller than the radius R of the drop ($h \ll R$). We wish to calculate the rate at which the plates will separate when we pull them apart with a perpendicular force F_{ext}. This force may for instance represent the weight of one plate in a horizontal configuration. We will neglect effects relating to wetting phenomena and surface tension.



1. What can be said about the velocity field from the symetries of the system ?
2. Write down the equations of motion of the fluid.
3. Integrate these equations using appropriate boundary conditions to obtain the velocity field $u_r(y)$. Comment on it. Then calculate the radial flow rate $Q(r)$.
4. By considering the flux through the boundaries of a volume element between r and $r + dr$ in a time interval dt, show that Reynolds' equation holds:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial p}{\partial r}\right) = 12\eta\frac{\partial h}{\partial t}$$

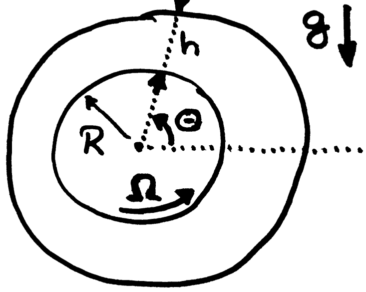
5. Integrate Reynolds' equation to find the pressure distribution (use the fact that the pressure cannot become infinitely large). Express the velocity using p. Does it depend on viscosity ? How does it depend on r ?
6. Taking the movement of the plate to be quasi-stationary, find the time evolution of the plate spacing $h(t)$ under the action of a constant external force F_{ext}. Note $h(t = 0) = h_0$ and identify a characteristic time τ in the evolution law.

*based on notes by Philippe Gondret

7. What external force has to be applied to the upper plate in order to make it descend at a velocity V, given the plate spacing h and the other parameters of the problem.
8. Check *a posteriori* that we were justified in neglecting certain terms in Navier-Stokes' equation.

2 The honey stick

When transferring honey from its glass to a slice of bread, one better turns the spoon or knive in order to maximise the quantity transferred and not to loose some on the way. Likewise meat grilled on a stick is turned so that it drips less into the fire. From a physical point of view, these systems may be modelled as a rotating cylinder coated with a liquid shell. [1].



Consider a cylinder of radius R turning horizontally about its axis at angular velocity Ω, entraining a layer of liquid of thickness $h(\theta, t)$ and viscous diffusivity $\nu = \eta/\rho$. We are interested in the steady state where the thickness of the liquid layer depends only on the angle θ with respect to the horizontal. We will treat this problem in the lubrication approximation, which means we will assume that intertial effects due to rotation remain small with respect to viscous effects, that the thickness h of the liquid shell is small compared to the cylinder radius R and that the film deformations remain small compared to its thickness. Furthermore we will neglect the viscosity of the surrounding air, as well as capillary phenomena. The gravitational acceleration will be noted g.

1. Write down and simplify the equations of motion. For each term that you neglect, specify the condition under which you are justified to do so.
2. Determine the velocity field in the liquid (use the notation $y = r - R$). Sketch the velocity field in $\theta = 0$, $\pi/2$, π et $3\pi/2$.
3. Obtain the expression for the flow rate $Q(\theta)$ per unit length of the cylinder.
4. Using the continuity equation in a slab of fluid, show that the evolution in the (weakly) non-stationary regime must obey:

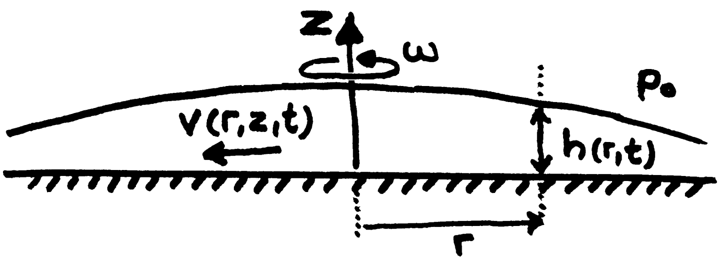
$$\frac{\partial h}{\partial t} = -\Omega \frac{\partial h}{\partial \theta} + \frac{g}{3\nu R} \frac{\partial}{\partial \theta} \left(h^3 \cos\theta \right).$$

1. This problem was proposed and solved by H. K. Moffatt in the 1970s: "Behaviour of a viscous film on the outer surface of a rotating cylinder," Journal de Mécanique **16**(5), pp. 651–673 (1977).

5. Show that, in the stationary case, thickness h and angle θ satisfy the following relation:
$$F(h) = \frac{g}{3\nu} h^3 \cos\theta - \Omega R h + Q = 0.$$
6. Argue why one cannot have a discontinuous film in the steady state. Graphically analyse the function $F(h)$ for particular values of θ ($\theta = \pm\pi/2$, 0, π). Show that in order to have a continuous stationary solution $h(\theta)$, the following condition must hold for the flow rate:
$$Q \leq \frac{2}{3}\Omega R \sqrt{\frac{\Omega R \nu}{g}}.$$
7. Deduce an order of magnitude for the maximum quantity of liquid that it is possible to maintain on a rotating cylinder (up to a numerical prefactor), given the parameters R, Ω, η, ρ and g of the system.

3 Spin coating

The "spin coating" technique is ubiquitous in the electronics industry, where it is used to coat solids with a thin liquid layer of controlled thickness. It consists in desspositing a thick layer of liquid on a flat substrate (for instance a disc-shaped silicon wafer), and then spinning the substrate at high angular velocity Ω. As it is entrained, most of the liquid is ejected by centrifugal forces, and the process is stopped when the desired thickness is reached.



In our model we will suppose that the flow can be locally decomposed into a radial velocity component of magnitude $u_r(r, z, t)$, and a tangential component $u_\theta(r)$ equal to the surface velocity of the substrate. Furthermore we assume that the layer is symmetric with respect to rotations about the axis $r = 0$, and has only weak surface slopes. Let p_0 be the atmospheric pressure above the liquid, and let ρ, η and $\nu = \eta/\rho$ be the density, viscosity and viscous diffusivity of the newtonian incompressible liquid. Neglecting gravity and capillary forces, we seek to describe the quasi-stationary regime.

1. What is the pressure distribution inside the liquid layer under the preceding assumptions ? What is the tangential velocity component $u_\theta(r)$ in the fixed laboratory frame of reference ?
2. State the equations of motion in the laboratory frame, in the lubrication approximation, and the appropriate boundary conditions. Determine the radial velocity profile $u_r(z)$.
3. Show that the local radial flow rate $q(r)$ is given by:
$$q = \frac{2\pi\Omega^2 r^2 h^3}{3\nu}$$

4. Using the mass conservation equation, show that

$$\frac{\partial h}{\partial t} = -\frac{\Omega^2}{\nu}\left(\frac{2}{3}h^3 + rh^2\frac{\partial h}{\partial r}\right). \tag{1}$$

5. Let us suppose that the liquid layer thickness is initially constant: $h(t=0) = h_0$. Rewrite equation (**??**) for this particular case, and integrate it to show that we have

$$h(t) = h_0\left(1 + \frac{t}{\tau}\right)^{-1/2}.$$

   Give the expression for the time constant τ. What is the asymptotic law for $h(t)$ as $t \gg \tau$?

6. What is the time constant τ for a rotation rate of 30 turns/s, and a liquid of viscosity 0.01 Pa s and mass density 10^3 kg/m^3 spread into a layer of initial thickness 0, 5 mm ? What is the film thickness after 10 s and 100 s of rotation, respectively ?

7. Calculate *a posteriori* at what times the non-linear term $u_r \partial u_r/\partial r$ and the unstationary term $\partial u_r/\partial t$ are indeed negligible compared to the viscous term $\nu\, \partial^2 u_r/\partial z^2$.