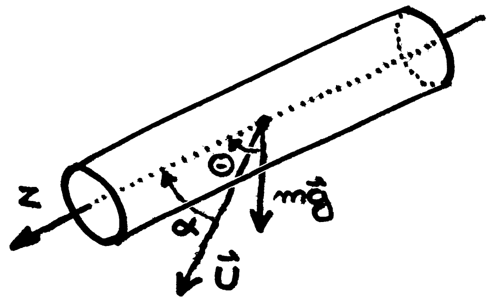
TD3 — Flow at low Reynolds numbers

M2 ICFP Physics of Fluids – Adrian Daerr*

1 Sedimentation of a stick

We would like to determine the trajectory of a stick falling through a viscous liquid in the steady state. We will model the stick by a cylinder, and note θ the angle by which its axis is tilted away from the vertical.



1. Given the symmetries of the problem, what can we say about the tensors A, B, C and D that link the forces and torques to the translational and rotational motion ? Defining friction coefficients $\lambda_\perp = A_{xx}$ and $\lambda_\parallel = A_{zz}$ perpendicular respectively parallel to the z-axis, write A, B and C out in full. Do we need to know the tensor D ?
2. For a cylinder of length L and radius R, one can show that

$$\lambda_\parallel = \frac{4\pi L}{\log(L/R) - 0,72} \quad \text{et} \quad \lambda_\perp = \frac{8\pi L}{\log(L/R) + 0,5}$$

   What is the ratio $\lambda_\perp/\lambda_\parallel$ for a very slender cylinder.
3. For a slender cylinder ($L \gg R$), calculate the relation between the angles θ and α, where α denotes the angles spanned by the trajectory and the axis of the cylinder. Specify the deviation $\theta-\alpha$ as a function of the cylinder tilt θ. Picture the trajectory of the falling cylinder. What is the maximum deviation, and for what value of θ does it occur ? To answer this it is convenient to consider the function $\tan(\theta - \alpha)$.

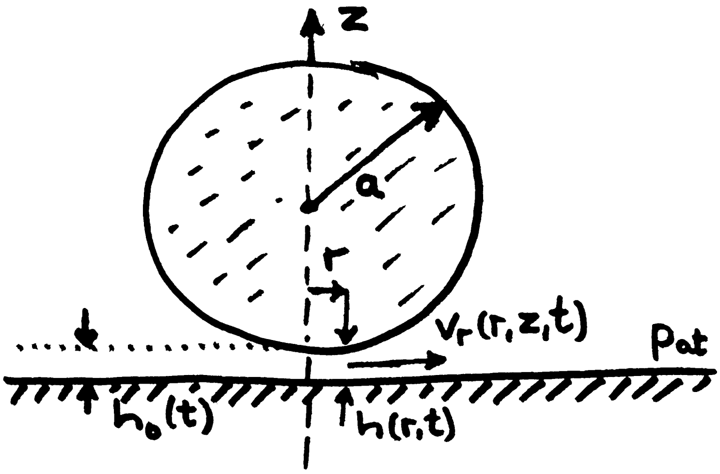
*based on notes by Philippe Gondret

2 Motion of a sphere approaching a plane

Let us analyse the motion of a rigid sphère of radius a and density ρ_s sedimenting under gravity towards an infinite solid horizontal plane, in an uncompressible newtonian fluid of mass density ρ and viscosity η (viscous diffusivity $\nu = \eta/\rho$. We will assume that the flow is laminar at all times, and we will neglect forces that are not of hydrodynamical origin, such as van der Waals and other short-ranged forces.

2.1 The smooth sphere

1. At first suppose that the sphere is perfectly smooth and far away from the plane. It sinks slowly enough for the Reynolds number to be much smaller than one. Recall the expression for the viscous force on the moving sphere, and calculate the vertical stationary velocity U_∞ as a function of the system parameters.
2. Suppose now that the sphere is very close to the wall, at a minimal distance h_0 already much less that its radius a. Suppose that we may apply the lubrication approximation, and that the local thickness $h(r,t)$ of the fluid layer at a distance r of the vertical symmetry axis is roughly $h(r,t) = h_0(t) + (r^2/2a)$.



a) Write down the mass conservation relation for the fluid between the sphere and the wall at a distance between r and $r + dr$ from the symmetry axis. Show that it can be written as

$$\frac{\partial h}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left(rh\overline{V_r}\right),$$

where $\overline{V_r}(r,t)$ is the average radial velocity at a distance r from the axis: $\overline{V_r}(r,t) = [1/h(r,t)] \int_0^{h(r,t)} V_r(r,z,t)\mathrm{d}z$.

b) Calculate the expression of the radial velocity $V_r(r,z,t)$ as a function of the pressure gradient $\partial p/\partial r$. Use it to rewrite the previous equation as follows:

$$\frac{\partial h}{\partial t} = \frac{1}{12\eta r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial p}{\partial r}\right),$$

c) Show that if p_{at} denotes the pressure outside the thin confined layer and near the wall, the pressure at a distance r where the fluid layer has a thickness h obeys the equation [1]

$$p = p_{\mathrm{at}} - \frac{3\eta a}{h^2}\frac{\partial h}{\partial t}$$

d) Conclude that the force resulting from the pressure distribution on the sphere goes as

$$F = -6\pi\eta \frac{a^2}{h_0}\frac{dh_0}{dt}$$

(as previously consider h_0 very small)
Compare this value to the drag force acting on a sphere far from solid boundaries.

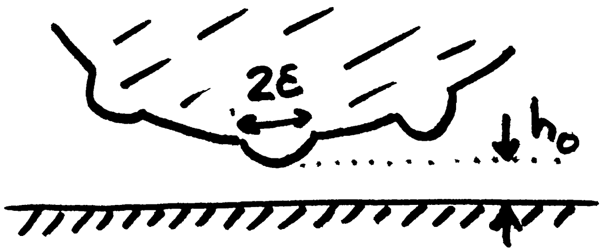
e) Write an equation of motion for the height dh_0/dt as the sphere sediments under its own weight (under what conditions may we assume that the mouvement is quasi-static ?). Calculate the height $h_0(t)$ given the initial distance h_{0i} at $t = 0$. (again h_0 and $h_{0i} \ll a$). Sketch $h_0(t)$. What is the characteristic time τ over which h_0 evolves ? When does the sphere touch the bottom ?

f) It is possible (though long and tedious) to show that for an arbitrary distance h_0 of the sphere from the bottom, the force acting on the sphere is given to a good approximation by the relation

$$F = -6\pi\eta \left(1 + \frac{a}{h_0}\right)\frac{dh_0}{dt}. \tag{1}$$

Argue why this result at least agrees with the two limits (?? and ??) found before.

2.2 The rough sphere



3. The aim of this part is to go one step further and to generalise these results to a simple way to estimate the roughness of a sphere sinking in a viscous liquid towards a horizontal bottom. The rugosity of the sphere of radius a will be modelled by a series of small hemispheres of radius ε attached to the original sphere (with $\varepsilon \ll a$, see figure). The distance $h_0(t)$ is still defined as the minimal distance between the (rough) sphere and the solid plane, at a given moment in time. We will consider the sphere to be in contact with the bottom once a number n of the hemispheres are closer than a small distance h_{0m} (typically a fraction of a nanometer) from the wall.

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1. Tip: use h instead of r as dependent variable for parts of the calculations.

a) Justify why $n = 3$ is a reasonable value when the sphere is rough enough. We will keep this value in the following.

b) Justify from equation (**??**) and from the argument above why the force F exerted by the fluid on the sphere approaching the bottom is approximated by

$$F = -6\pi\eta \left(1 + \frac{a}{h_0 + \varepsilon} + \frac{3\varepsilon^2}{a h_0}\right) \frac{dh_0}{dt}.$$

4. We now turn the system upside-down, keeping the sphere in contact with the solid wall (still immersed in the liquid). At time $t = 0$ we allow the sphere to fall away from the wall under the action of gravity.

a) Can the previous results (**??** to **??**) be re-used to solve this problem ?

b) What differential equation does $h_0(t)$ obey ? Show that $h_0(t)$ satisfies the following implicit equation:

$$h_0 - h_{0m} + a \log\left(\frac{h_0 + \varepsilon}{h_{0m} + \varepsilon}\right) + \frac{3\varepsilon^2}{a} \log\left(\frac{h_0}{h_{0m}}\right) = \frac{2(\rho_x - \rho) g a^2}{9\eta} t,$$

where h_{0m} is the initial distance between the wall and the asperities of the rough sphere. Evaluate the various terms to show that for typical values of the rugosity (a few microns) and values h_{0m} of the order of atomic distances ($0.5 \cdot 10^{-9}$ m), the influence of h_{0m} (as well as that of n) on the falling time is negligible compared to the other terms.

c) The measurement of the falling time starting from the wall thus leads to an experimental estimate of the relative rugosity ε/a of the sphere as a function of the other parameters of the problem. The role of the other parameters can be eliminated by performing two successive measurements of the falling time. Show in particular that the relative rugosity can be estimated from the times t_a and t_{2a} it takes the sphere to cover distances of a respectively $2a$ from its position at contact, according to the formula

$$\frac{\varepsilon}{a} = 2 \exp\left(2 - \frac{t_{2a}(1 + \log 2)}{t_{2a} - t_a}\right).$$

d) Estimate the times t_a and t_{2a} (and the ratio t_{2a}/t_a) for beads of diameter $2a = 1\,\text{mm}$, density $\rho_s = 1050\,\text{kg/m}^3$, and roughness $\varepsilon = 1\,\mu\text{m}$ or $\varepsilon = 10\,\mu\text{m}$. The liquid has density $\rho = 1000\,\text{kg/m}^3$ and viscosité $\eta = 10^{-2}\,\text{Pa s}$ or $10^{-1}\,\text{Pa s}$, and $g = 10\,\text{m/s}^2$.