«Boundary Layer Theory», H. Schlichting, K. Gersten, 8th edition (2000), Springer Berlin Heidelberg New York, ISBN 978-3-540-66270-9

7.5 Asymptotic Behaviour of Solutions Downstream

In what follows we shall investigate the asymptotic behaviour of boundary–layer solutions far downstream. Here again we deal with series expansions of the solutions, but in this case for large values of x. The main term under consideration in this expansion is the leading term which reflects the asymptotic behaviour of the solution for $x \to \infty$.

7.5.1 Wake Behind Bodies

As the examples of the mixing layer and the free jet showed, the application of the boundary–layer equations is not necessarily restricted to the presence of fixed walls. They can also be applied when there is a layer with dominating frictional effects in the interior of a flow.

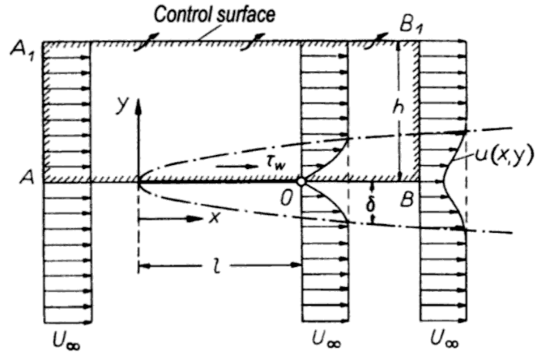

Fig. 7.9. Wake behind a two–dimensional body

The wake flow behind a flat plate of length l as in Fig. 7.9 is also such an example. The two boundary layers on the upper and lower sides coalesce at the trailing edge and further downstream produce a *wake profile*, whose width increases with increasing distance from the body, while the *velocity defect* decreases. On the whole, as we will see later, the shape of the velocity profile in the wake, also called *wind shadow*, is independent of the shape of the body for $x \to \infty$, up to a scaling factor. The asymptotic development for $x \to \infty$ has been given by W. Tollmien (1931). Since the magnitude of the velocity defect continually decreases with increasing x, it can be assumed for $x \to \infty$ that the velocity defect

$$u_1(x,y) = U_\infty - u(x,y) \tag{7.86}$$

is small compared to U_∞, so that quadratic terms of u_1 and equivalently of v_1 can be neglected. Since the pressure is constant far downstream, we use the boundary–layer equation (7.4) and insert Eq. (7.86), neglecting the quadratic terms in u_1 and v_1 to obtain

$$U_\infty \frac{\partial u_1}{\partial x} = \nu \frac{\partial^2 u_1}{\partial y^2} \tag{7.87}$$

with the boundary conditions

$$y = 0: \quad \frac{\partial u_1}{\partial y} = 0; \qquad y \to \infty: \quad u_1 = 0\,.$$

This equation is a *linear* partial differential equation. This linearity is characteristic for the computation of *small perturbations*. The differential equation is, as Eq. (7.80), again identical to the unsteady heat conduction equation. With the trial solution

$$u_1 = U_\infty C \left(\frac{x}{l}\right)^{-m} F(\eta)\,, \quad \eta = \frac{y}{2}\sqrt{\frac{U_\infty}{\nu x}} \tag{7.88}$$

we obtain the following differential equation for the function $F(\eta)$:

$$F'' + 2\eta F' + 4mF = 0 \tag{7.89}$$

with the boundary conditions

$$\eta = 0: \quad F' = 0; \qquad \eta \to \infty: \quad F = 0\,.$$

Table 7.2. Balance of volume flux and x momentum on the control surface in Fig. 7.9

cross–section	volume flux	x momentum
AB	0	0
AA_1	$b\int_0^h U_\infty\,dy$	$\varrho b\int_0^h U_\infty^2\,dy$
BB_1	$-b\int_0^h u\,dy$	$-\varrho b\int_0^h u^2\,dy$
A_1B_1	$-b\int_0^h (U_\infty - u)\,dy$	$-\varrho b\int_0^h U_\infty(U_\infty - u)\,dy$
$\sum$ = control surface	$\sum$ volume flux = 0	$\sum$ momentum flux = drag

The still unknown exponent m (*eigenvalue*) can be determined via a global momentum balance around the body in Fig. 7.9. The rectangular control surface AA_1B_1B is placed far enough away from the body that the pressure on it is unperturbed. The pressure is constant over the whole of the control surface and so there is no contribution to the momentum balance from the pressure forces. In calculating the momentum flux across the control surface, we must note that for continuity reasons fluid must flow out across the upper and lower surfaces. The quantity of fluid leaving through A_1B_1 is equal to the difference between that entering through AA_1 and leaving through BB_1. The momentum balance is given in Table 7.2, where inflowing volume fluxes are counted positive and outflowing volume fluxes negative. Then the drag corresponds to the total momentum flux, thus giving us

$$D = b\varrho \int_{-\infty}^{+\infty} u(U_\infty - u)\,dy\,. \tag{7.90}$$

Here the limits of integration may be set to $y = \pm\infty$ instead of $y = \pm h$, since the integrand in Eq. (7.90) vanishes for $|y| > h$. With the trial solution (7.88), Eq. (7.90) yields

$$D \approx b\varrho \int_{-\infty}^{+\infty} U_\infty u_1\,dy = 2b\varrho U_\infty^2 C \left(\frac{x}{l}\right)^{-m} \sqrt{\frac{\nu x}{U_\infty}} \int_{-\infty}^{+\infty} F(\eta)\,d\eta\,. \tag{7.91}$$

Since this balance must be independent of x, it follows that $m = 1/2$. Equation (7.89) thus determined becomes

$$F'' + 2\eta F' + 2F = 0 \tag{7.92}$$

which, after integrating once, yields

$$F' + 2\eta F = 0$$

with the solution

$$F(\eta) = e^{-\eta^2} \,. \tag{7.93}$$

Using the integral

$$\int_{-\infty}^{+\infty} F(\eta)\, d\eta = \int_{-\infty}^{+\infty} e^{-\eta^2}\, d\eta = \sqrt{\pi}$$

it follows from Eq. (7.91) that the drag coefficient is

$$c_D = \frac{D}{\frac{\varrho}{2} U_\infty^2 bl} = \frac{4\sqrt{\pi} C}{\sqrt{\frac{U_\infty l}{\nu}}} \,. \tag{7.94}$$

Therefore the final solution for the defect velocity in the wake of a body with drag coefficient c_D is

$$\frac{u_1(x,y)}{U_\infty} = \frac{c_D}{4\sqrt{\pi}} \sqrt{\frac{U_\infty l}{\nu}} \left(\frac{x}{l}\right)^{-\frac{1}{2}} \exp\left(-\frac{y^2 U_\infty}{4x\nu}\right) \,. \tag{7.95}$$

From Eq. (7.88) it then follows that the half–value width of the wake is

$$y_{0.5} = 1.7 \sqrt{\frac{\nu x}{U_\infty}} \tag{7.96}$$

i.e. here too the width of the frictional layer is proportional to $\sqrt{\nu}$.

It is worth noting that, in spite of the widening of the wake, the defect volume flux in the wake is independent of x, i.e. no side entrainment occurs in this flow. The compensating volume flux which flows out through the sides of the control surface does so already in the *near field* of the body, and not in the *far field* described by the solution (7.95). This solution may be used at about $x > 3l$.

For the extension of this solution to smaller x values, see the work by S.A. Berger (1971), p. 237.

In most practical cases, wake flows are turbulent, since the velocity profiles in the wake possess points of inflection and are thus particularly unstable. Consequently the transition to turbulent flow takes place at relatively low Reynolds numbers, cf. Chap. 15.

Note (Jet in parallel flow)

The wake solution is also valid for the asymptotic decay of a free jet flow in an equally directed parallel flow. Instead of the drag coefficient c_D, the analogously defined *jet momentum coefficient* c_μ appears, and $u_1(x,y)$ is interpreted as an excess velocity.