Nonlinear Spectral Synthesis of Soliton Gas in Deep-Water Surface Gravity Waves

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Soliton gases represent large random soliton ensembles in physical systems that exhibit integrable dynamics at the leading order. Despite significant theoretical developments and observational evidence of ubiquity of soliton gases in fluids and optical media, their controlled experimental realization has been missing. We report a controlled synthesis of a dense soliton gas in deep-water surface gravity waves using the tools of nonlinear spectral theory [inverse scattering transform (IST)] for the one-dimensional focusing nonlinear Schrödinger equation. The soliton gas is experimentally generated in a one-dimensional water tank where we demonstrate that we can control and measure the density of states, i.e., the probability density function parametrizing the soliton gas in the IST spectral phase space. Nonlinear spectral analysis of the generated hydrodynamic soliton gas reveals that the density of states slowly changes under the influence of perturbative higher-order effects that break the integrability of the wave dynamics.

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Solitons are localized nonlinear waves that have been studied in many areas of science over the last decades [1–3]. Solitons represent fundamental nonlinear modes of physical systems described by integrable equations like the Korteweg–de Vries (KdV) equation or the one-dimensional nonlinear Schrödinger equation (1D-NLSE) [1,3–8]. Nowadays the dynamics of soliton interaction is so well mastered that ordered sets of optical solitons or their periodic generalizations, the so-called finite-gap potentials, are synthesized and manipulated to carry out the transmission of information in fiber optics communication links [9–12]. On the other hand, the question of collective dynamics of *large random* soliton ensembles represents a subject of active research in statistical mechanics and in nonlinear physics, most notably in the contexts of ocean wave dynamics and nonlinear optics, see, e.g., Refs. [13–25].

The concept of soliton gas (SG) as a large ensemble of solitons randomly distributed on an infinite line and elastically interacting with each other originates from the work of Zakharov [26], who introduced the kinetic equation for a nonequilibrium *diluted* gas of weakly interacting solitons of the KdV equation. Zakharov's kinetic equation has been generalized to the case of a dense SG in Ref. [27] (KdV) and in Refs. [28,29] (focusing NLS). Additionally, the SG kinetic equation has recently attracted much attention in the context of generalized hydrodynamics for quantum many-body integrable systems; see Refs. [30–32] and references therein.

Within the inverse scattering transform (IST) formalism [5,7,33], each soliton in a gas is characterized by a discrete eigenvalue λ_i of the spectrum of the linear operator associated with the integrable evolution equation. The fundamental property of integrable dynamics is the preservation of the soliton spectrum under evolution. The central concept in SG theory, borrowed from quantum physics of disordered systems [34], is the density of states (DOS), which represents the distribution $u(\lambda, x, t)$ over the spectral eigenvalues, so that $u d\lambda dx$ is the number of soliton states found at time t in the element of the phase space $[\lambda, \lambda + d\lambda] \times [x, x + dx]$. In a spatially homogeneous (equilibrium) SG, the DOS is stationary in time, while in a nonhomogeneous SG, which is far from equilibrium, it evolves according to the kinetic equation [27–29].

Despite various developments of SG theory (see, e.g., Refs. [35–42]) and the existence of an unambiguous characterization of SG through the concept of DOS, the experimental or observational results in this area are quite limited. Costa *et al* have reported in 2014 the observation of random wave packets in shallow water ocean waves that have been analyzed using numerical IST tools and interpreted as randomly distributed solitons that might be associated with KdV SG [43]. In 2015 large ensembles of interacting and colliding solitons have been observed in a levitating rectilinear water cylinder [44]. Recently Redor *et al.* have used the process of fission of a sinusoidal wave train to generate an ensemble of bidirectional shallow water solitons in a 34-m long flume and they have analyzed the generated SG using (linear) Fourier transform [45]. In optics, the SG terminology has been used to describe experiments where light pulses were synchronously injected in a passive fiber ring cavity [46]. Another recent experimental observation of complex nonlinear wave behavior attributed to SG dynamics was reported in Ref. [47], where the formation of an incoherent optical field has been observed in the evolution of a square pulse in a focusing medium [48]. At the same time, to our knowledge, there is no existing experiment where SG would have been unambiguously identified using some well-defined, quantitative characteristics and, in particular, where the measurement and control of the DOS of the SG would have been achieved.

In this Letter, we report experiments fully based on the IST method where we generate and study hydrodynamic deep-water dense and spatially homogeneous SGs. We take advantage of the recently developed methodology for the effective numerical construction of the so-called N-soliton solutions of the focusing 1D-NLSE with N large (Ref. [49]), to create an incoherent wave field having a dominant and controlled solitonic content characterized by a measurable DOS. We show that the nonlinear wave field in the generated SG may undergo some complex space-time evolution while the discrete IST spectrum is found to be nearly conserved, albeit being perturbed by higher-order effects.

Our experiments were performed in a wave flume 148 m long, 5 m wide, and 3 m deep. Unidirectional waves are generated at one end with a computer assisted flap-type wave maker and the flume is equipped with an absorbing device strongly reducing wave reflection at the opposite end. As in Ref. [50], the setup comprises 20 equally spaced resistive wave gauges that are installed along the basin at distances $Z_j = j \times 6$ m, $j = 1, 2, \ldots, 20$ from the wave maker located at $Z = 0$ m. This provides an effective measuring range of 114 m.

In our experiment, the water elevation at the wave maker reads $\eta(Z = 0, T) = \mathrm{Re}[A_0(T) e^{i\omega_0 T}]$, where $\omega_0 = 2\pi f_0$ is the angular frequency of the carrier wave. $A_0(T)$ represents the complex envelope of the initial condition. Our experiments are performed in the deep-water regime, and they are designed in such a way that the observed dynamics is described at leading order by the focusing 1D-NLSE:

$$\frac{\partial A}{\partial Z} + \frac{1}{C_g}\frac{\partial A}{\partial T} = i\frac{k_0}{\omega_0^2}\frac{\partial^2 A}{\partial T^2} + i\alpha k_0^3 |A|^2 A, \qquad (1)$$

where $A(Z, T)$ represents the complex envelope of the water wave that changes in space Z and in time T [51]. k_0 represents the wave number of the propagating wave $[\eta(Z, T) = \mathrm{Re}[A(Z, T) e^{i(\omega_0 T - k_0 Z)}]]$, which is linked to ω_0 according to the deep water dispersion relation $\omega_0^2 = k_0 g$, where g is the gravity acceleration. $C_g = g/(2\omega_0)$ represents the group velocity of the wave packets and α is a dimensionless term describing the small finite-depth correction to the cubic nonlinearity [50].

The first important step of the experiment consists in generating an initial condition $A_0(T)$ in the form of a random wave field having a pure solitonic content. To achieve this, we move to the “IST-friendly” canonical dimensionless form of the 1D-NLSE:

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = 0, \qquad (2)$$

where $\psi(x, t)$ represents the normalized complex envelope of the water wave. Connections between the physical variables of Eq. (1) and dimensionless variables in Eq. (2) are given by $t = Z/L_{\mathrm{NL}}$, $x = (T - Z/C_g)\sqrt{g/(2L_{\mathrm{NL}})}$ with the nonlinear length being defined as $L_{\mathrm{NL}} = 1/(\alpha k_0^3 \langle |A_0(T)|^2 \rangle)$, where the angle brackets denote the average over time.

The nonlinear wave field $\psi(x, t)$ satisfying Eq. (2) can be characterized by the so-called scattering data (the IST spectrum). For the *localized*, i.e., decaying to zero as $|x| \to \infty$ wave field the IST spectrum consists of a discrete part related to the soliton content and a continuous part related to the dispersive radiation. A special class of solutions, the N-soliton solutions (N-SSs), exhibit only a discrete spectrum consisting of N complex-valued eigenvalues λ_n, $n = 1, \ldots, N$ and N complex parameters $C_n = |C_n| e^{i\phi_n}$, called norming constants, defined for each λ_n. In all the experiments described below, the phases ϕ_n of the norming constants C_n characterizing the generated N-SS are randomly and uniformly distributed over $[0, 2\pi)$, while their modulus $|C_n|$ is chosen to be equal to unity. As shown in Refs. [49,52], such an N-soliton statistical ensemble is a good model for a homogeneous dense SG.

In our first experimental run, we used numerical methods described in Ref. [49] to generate an N-SS of Eq. (2), hereafter denoted $\psi_{16}(x, t)$, with $N = 16$ eigenvalues chosen arbitrarily within some domain of the complex spectral plane, as shown with blue points in Fig. 1(b). A relatively small number of solitons in this random soliton ensemble prevents its proper macroscopic spectral characterization and the identification with SG. However, it is important as a first step in our experiment to establish a robust protocol for the generation of random soliton ensembles in a spectrally controlled way.

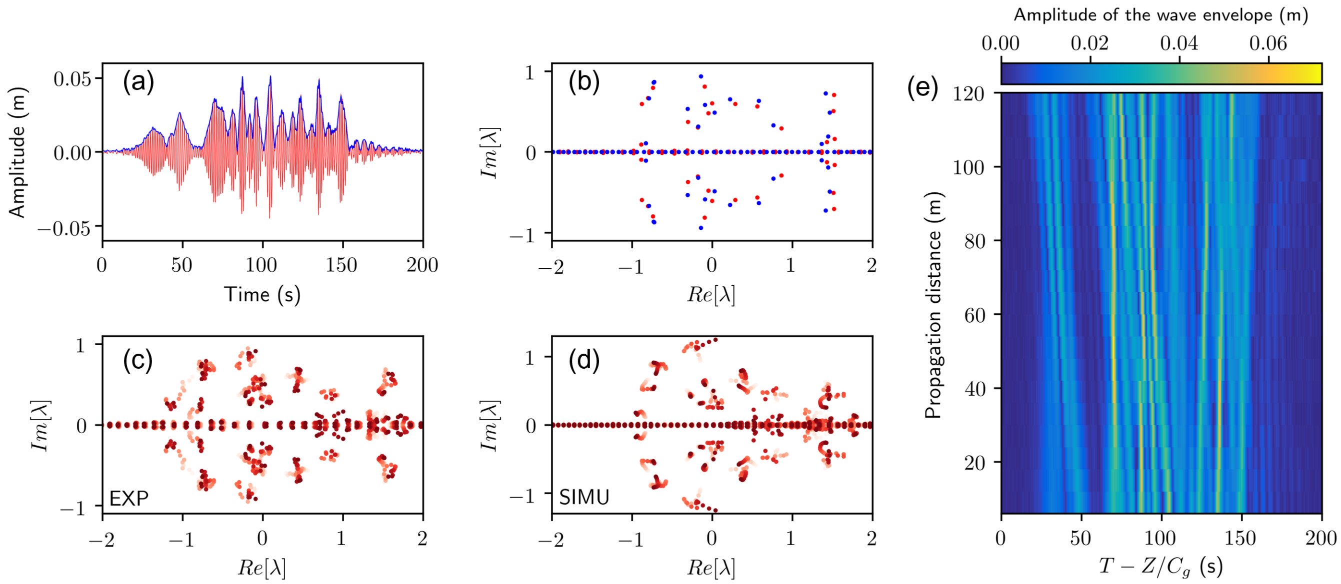

FIG. 1. Ensemble of $N = 16$ solitons propagating in the 1D water tank. (a) Water elevation (red line) and modulus of the wave envelope measured at $Z_1 = 6$ m, close to the wave maker. (b) Blue points represent the discrete IST spectrum of the numerically generated N-SS $\psi_{16}(x, t = 0)$ and red points represent the discrete IST spectrum measured at $Z_1 = 6$ m by using the signal plotted in (a). (c) Space evolution of the discrete IST spectra measured along the tank from $Z_1 = 6$ (light red) to $Z_{20} = 120$ m (dark red). (d) Same as in (c) but obtained from numerical simulations of a modified (not integrable) 1D-NLSE including higher-order effects; see Supplemental Material [53]. (e) Space-time evolution of modulus of the wave envelope recorded by the 20 gauges regularly spaced along the tank. Physical parameters characterizing the experiment are $f_0 = 0.9$ Hz, $k_0 = 3.26$ m^{-1}, $\alpha = 0.895$, $L_{\rm NL} = 210$ m ($\langle |A_0(T)|^2 \rangle = 1.53 \times 10^{-4}$ m^2).

After appropriate scaling, the generated dimensionless wave field $\psi_{16}(x, t = 0)$ is converted into the physical complex envelope $A_{16}(Z = 0, T) = A_0(T)$ of the initial condition, which is generated by the wave maker. Figure 1(a) shows the water elevation measured at $Z_1 = 6$ m together with the modulus of the envelope $|A_{16}(Z_1, t)|$ computed using standard Hilbert transform techniques [51]. The generated wave field with pure solitonic content spreads over approximately 140 s and exhibits large amplitude fluctuations due to the random phase distribution. Figure 1(b) shows the discrete IST spectrum that is computed from the signal recorded by the first gauge and plotted in Fig. 1(a). The measured eigenvalues plotted in red points in Fig. 1(b) are close to the discrete eigenvalues (blue points) that we have selected to build $\psi_{16}(x, t = 0)$, the N-SS under consideration. This demonstrates that the process of generation of the N-SS solution is well controlled in our experiments.

As shown in Fig. 1(e), the space-time evolution of the generated wave packet measured with 20 gauges distributed along the tank reveals complex dynamics with multiple interacting coherent structures. Despite the apparent complexity of the observed wave evolution, the measured discrete IST spectra, compiled and superimposed in Fig. 1(c), are nearly conserved over the whole propagation distance.

The fact that the isospectrality condition perfectly fulfilled in a numerical simulation of the 1D-NLSE (see Supplemental Material [53]) is not exactly verified in the experiment arises from perturbative higher-order effects that break the integrability of the wave dynamics [50,61,62]. As shown in Fig. 1(d), numerical simulations of a modified NLSE including higher-order effects (see Supplemental Material [53]) reveals that each discrete eigenvalue follows an individual trajectory in the complex plane under the influence of higher-order effects. In the experiment, these trajectories are not resolved because of measurement inaccuracies; compare Fig. 1(c) and Fig. 1(d). Nevertheless, the results of the nonlinear spectral analysis reported in Fig. 1(c) show that the dynamical features observed for the wave field composed of 16 solitons are nearly integrable.

We now take advantage of the above method of the controlled generation of multiple-soliton, random phase solutions of the 1D-NLSE to generate a random N-soliton ensemble that can be identified as SG. It is clear from the definition of DOS in SG theory that to achieve that, the number of solitons N should be sufficiently large. Figure 2 shows the dynamical and spectral features characterizing the experimental evolution of an ensemble of $N = 128$ solitons with random spectral (IST) characteristics. The important difference with the first example is that, due to a large number of solitons generated, we are now able to characterize the soliton ensemble by a DOS $u(\lambda)$; see Fig. 3. Specifically, we generate a SG with eigenvalues $\lambda_i \in \mathbb{C}$ distributed nearly uniformly on a rectangle in the upper half-plane of the complex IST spectral plane (and the

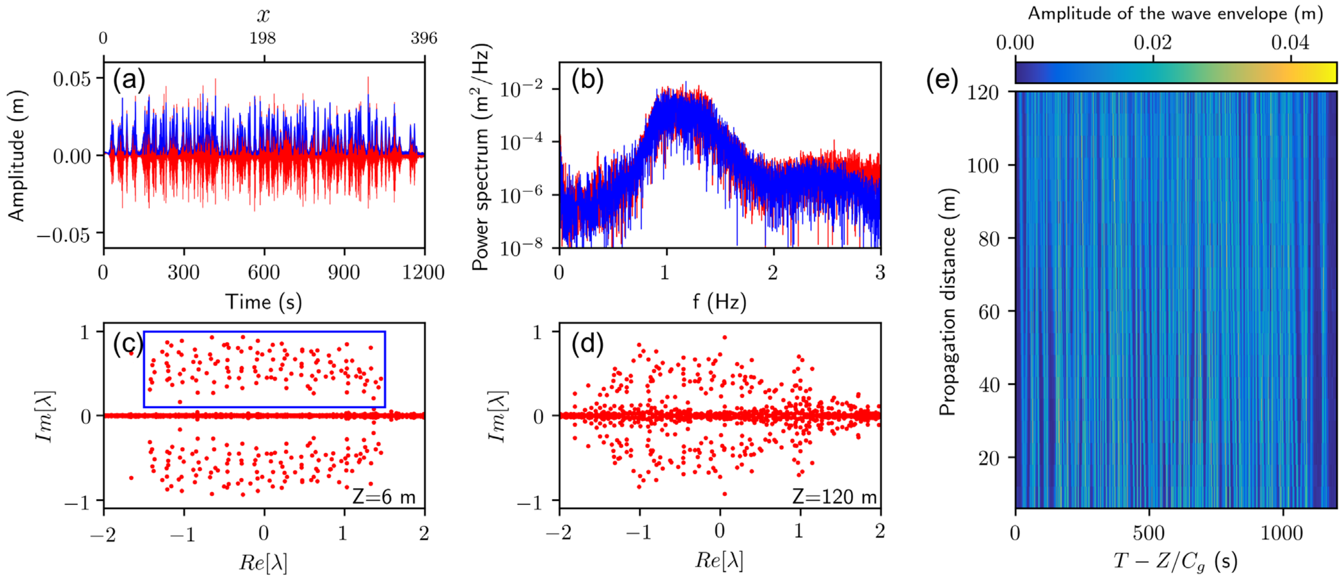

FIG. 2. Gas of $N = 128$ solitons propagating in the 1D water tank. (a) Water elevation (red line) and modulus of the wave envelope measured at $Z_1 = 6$ m, close to the wave maker. (b) Fourier power spectra of wave elevation at $Z_1 = 6$ (blue line) and at $Z_{20} = 120$ m (red line). (c) Discrete IST spectrum measured at $Z_1 = 6$ m. (d) Discrete IST spectrum measured at $Z_{20} = 120$ m. (e) Space-time evolution of modulus of the wave envelope recorded by the 20 gauges regularly spaced along the tank. Physical parameters characterizing the experiment are $f_0 = 1.15$ Hz, $k_0 = 5.32$ m^{-1}, $\alpha = 0.936$, $L_{\rm NL} = 45$ m ($\langle|A_0(T)|^2\rangle = 1.58 \times 10^{-4}$ m^2).

c.c. rectangle in the lower half plane) and the DOS $u(\lambda) = u_0$ being nearly constant within the rectangle, see Fig. 2(c). In other words, we generate a (approximately) *homogeneous* SG.

Figure 2(a) shows that the generated SG has the form of a random wave field spreading over $\Delta T = 1200$ s which corresponds to a range $\Delta x = 396$ in the dimensionless variables of Eq. (2). Clearly the generated SG does not represent a *diluted* SG composed of isolated and weakly interacting solitons but rather a *dense* SG which cannot be represented as superposition of individual solitons. Figure 2(b) shows that the propagation of the generated SG is not accompanied by any significant broadening of Fourier power spectrum.

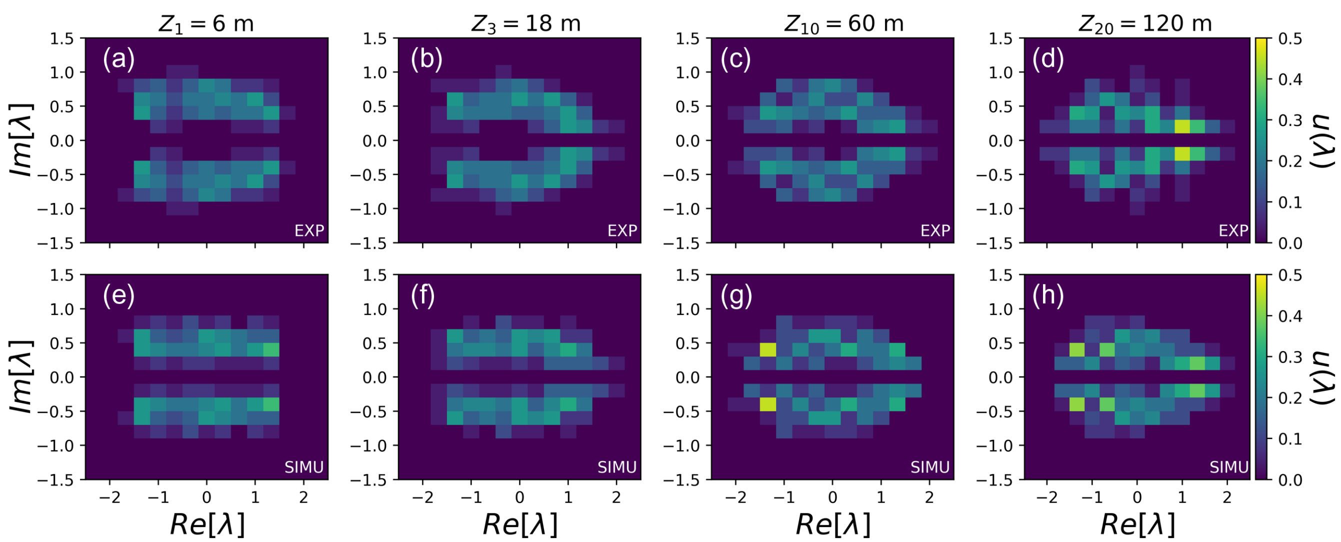

FIG. 3. Statistical analysis of discrete IST spectra of the gas of 128 solitons showing the slow evolution of the DOS $u(\lambda)$ (the probability density function of the discrete IST eigenvalues in the complex plane) as a function of propagation distance in the water tank: (a),(e) $z_1 = 6$; (b),(f) $z_3 = 18$; (c), (g) $z_{10} = 60$; (d), (h) $z_{20} = 120$ m. The upper row (a)–(d) represents the DOS measured in the experiment while the lower row represents the DOS computed in numerical simulation of Euler's equations; see Supplemental Material for details [53].

Figure 2(c) shows the discrete IST spectrum of the wave field measured at $Z_1 = 6$ m. A set of $N = 128$ eigenvalues is now measured within a rectangle in the upper complex plane. Similarly to the features reported in Fig. 1, the perturbative higher-order effects influence the observed dynamics and the discrete spectrum measured at $Z_{20} = 120$ m is not identical to the one measured at Z_1, see Fig. 2(d). Even though the isospectrality condition characterizing a purely integrable dynamics is not exactly satisfied in our experiment, the measured discrete spectrum remains confined to a well-defined region of the complex plane. Moreover, the large number of eigenvalues distributed with some density within this limited region of the complex plane justifies the introduction of a statistical description of the spectral (IST) data, which represents the key point for the analysis of the observed wavefield in the framework of the SG theory.

In the context of the 1D-NLSE (2) the DOS $u(\lambda)$, where $\lambda = \beta + i\gamma$, represents the density of soliton states in the phase space, i.e., $ud\beta d\gamma dx$ is the number of solitons contained in a portion of SG with the complex spectral parameter $\lambda \in [\beta, \beta + d\beta] \times [\gamma, \gamma + d\gamma]$ over the space interval $[x, x + dx]$ at time t (corresponding to the position Z in the tank). Considering that the generated SG is homogeneous in space, the DOS represents the probability density function of the complex-valued discrete eigenvalues normalized in such a way that $\int_{-\infty}^{+\infty} d\beta \int_0^{+\infty} d\gamma u(\lambda) = N/\Delta x$, where N represents the number of eigenvalues found in the upper complex plane and Δx represents the spatial extent of the gas. Figure 3 (upper row) displays the normalized DOS experimentally measured at different propagation distances in the water tank while Fig. 3 (lower row) displays the normalized DOS computed in a numerical simulation of Euler's equations; see Supplemental Material [53] for details. The experiments and numerical simulations reveal a slow evolution of the DOS along the tank occurring over a characteristic length scale determined by $L_{\mathrm{NL}} \Delta f / f_0$, where Δf represents the spectral bandwidth of the wave field (see Supplemental Material [53]). This slow evolution is not due to the gas nonhomogeneity, but mainly originates from the presence of perturbative higher-order effects. Results reported in Fig. 3 suggest that the incorporation of higher-order perturbative physical effects in the theory of SG represents a theoretical question of significant interest.

In this Letter, we have reported hydrodynamic experiments demonstrating that a controlled synthesis of a dense SG can be achieved in deep-water surface gravity waves. We show that the generated homogeneous SG is characterized by a measurable spectral DOS, which provides an essential first step towards experimental verification of the kinetic theory of nonequilibrium SGs. We hope that our work will stimulate new experimental and theoretical research in the fields of statistical mechanics and nonlinear random waves.

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[1] M. Remoissenet, *Waves Called Solitons: Concepts and Experiments*, 2nd ed. (Springer, Berlin, 1996).
[2] Y. V. Kartashov, B. A. Malomed, and L. Torner, Rev. Mod. Phys. **83**, 247 (2011).
[3] T. Dauxois and M. Peyrard, *Physics of Solitons* (Cambridge University Press, Cambridge, England, 2006).
[4] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).
[5] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method* (Springer Science Business Media, New York, 1984).
[6] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **31**, 125 (1973).
[7] J. Yang, *Nonlinear Waves in Integrable and Non-integrable Systems*, Mathematical Modeling and Computation (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2010).
[8] S. Trillo, G. Deng, G. Biondini, M. Klein, G. F. Clauss, A. Chabchoub, and M. Onorato, Phys. Rev. Lett. **117**, 144102 (2016).
[9] S. T. Le, V. Aref, and H. Buelow, Nat. Photonics **11**, 570 (2017).
[10] S. K. Turitsyn, J. E. Prilepsky, S. T. Le, S. Wahls, L. L. Frumin, M. Kamalian, and S. A. Derevyanko, Optica **4**, 307 (2017).
[11] S. T. Le, J. E. Prilepsky, and S. K. Turitsyn, Opt. Express **22**, 26720 (2014).
[12] M. Kamalian, A. Vasylchenkova, D. Shepelsky, J. E. Prilepsky, and S. K. Turitsyn, J. Lightwave Technol. **36**, 5714 (2018).
[13] A. R. Osborne and T. L. Burch, Science **208**, 451 (1980).
[14] A. R. Osborne, Phys. Rev. E **52**, 1105 (1995).
[15] M. Onorato, A. R. Osborne, M. Serio, and S. Bertone, Phys. Rev. Lett. **86**, 5831 (2001).
[16] M. Onorato, S. Residori, U. Bortolozzo, A. Montina, and F. Arecchi, Phys. Rep. **528**, 47 (2013).

[17] E. Pelinovsky, C. Kharif *et al.*, *Extreme Ocean Waves* (Springer, New York, 2008).
[18] M. Onorato, A. R. Osborne, M. Serio, and L. Cavaleri, Phys. Fluids **17**, 078101 (2005).
[19] R. Hassaini and N. Mordant, Phys. Rev. Fluids **2**, 094803 (2017).
[20] R. El Koussaifi, A. Tikan, A. Toffoli, S. Randoux, P. Suret, and M. Onorato, Phys. Rev. E **97**, 012208 (2018).
[21] S. Randoux, P. Walczak, M. Onorato, and P. Suret, Phys. Rev. Lett. **113**, 113902 (2014).
[22] Y. Bromberg, U. Lahini, E. Small, and Y. Silberberg, Nat. Photonics **4**, 721 (2010).
[23] J. M. Soto-Crespo, N. Devine, and N. Akhmediev, Phys. Rev. Lett. **116**, 103901 (2016).
[24] J. M. Dudley, F. Dias, M. Erkintalo, and G. Genty, Nat. Photonics **8**, 755 (2014).
[25] A. E. Kraych, D. Agafontsev, S. Randoux, and P. Suret, Phys. Rev. Lett. **123**, 093902 (2019).
[26] V. E. Zakharov, Sov. Phys. JETP **33**, 538 (1971).
[27] G. El, Phys. Lett. A **311**, 374 (2003).
[28] G. A. El and A. M. Kamchatnov, Phys. Rev. Lett. **95**, 204101 (2005).
[29] G. El and A. Tovbis, Phys. Rev. E **101**, 052207 (2020).
[30] B. Doyon, T. Yoshimura, and J.-S. Caux, Phys. Rev. Lett. **120**, 045301 (2018).
[31] B. Doyon, H. Spohn, and T. Yoshimura, Nucl. Phys. **B926**, 570 (2018).
[32] D.-L. Vu and T. Yoshimura, SciPost Phys. **6**, 023 (2019).
[33] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
[34] I. Lifshits, S. Gredeskul, and L. Pastur, *Introduction to the Theory of Disordered Systems* (Wiley, New York, 1988).
[35] J. D. Meiss and W. Horton, Jr., Phys. Rev. Lett. **48**, 1362 (1982).
[36] A. Fratalocchi, A. Armaroli, and S. Trillo, Phys. Rev. A **83**, 053846 (2011).
[37] G. A. El, A. M. Kamchatnov, M. V. Pavlov, and S. A. Zykov, J. Nonlinear Sci. **21**, 151 (2011).
[38] D. Dutykh and E. Pelinovsky, Phys. Lett. A **378**, 3102 (2014).
[39] F. Carbone, D. Dutykh, and G. A. El, Europhys. Lett. **113**, 30003 (2016).
[40] E. Shurgalina and E. Pelinovsky, Phys. Lett. A **380**, 2049 (2016).
[41] M. Girotti, T. Grava, and K. D. T.-R. McLaughlin, arXiv: 1807.00608.
[42] D. Kachulin, A. Dyachenko, and A .Gelash, Fluids **4**, 83 (2019).
[43] A. Costa, A. R. Osborne, D. T. Resio, S. Alessio, E. Chrivì, E. Saggese, K. Bellomo, and C. E. Long, Phys. Rev. Lett. **113**, 108501 (2014).
[44] S. Perrard, L. Deike, C. Duchêne, and C.-T. Pham, Phys. Rev. E **92**, 011002(R) (2015).
[45] I. Redor, E. Barthélemy, H. Michallet, M. Onorato, and N. Mordant, Phys. Rev. Lett. **122**, 214502 (2019).
[46] A. Schwache and F. Mitschke, Phys. Rev. E **55**, 7720 (1997).
[47] G. Marcucci, D. Pierangeli, A. J. Agranat, R.-K. Lee, E. DelRe, and C. Conti, Nat. Commun. **10**, 5090 (2019).
[48] G. A. El, E. G. Khamis, and A. Tovbis, Nonlinearity **29**, 2798 (2016).
[49] A. A. Gelash and D. S. Agafontsev, Phys. Rev. E **98**, 042210 (2018).
[50] F. Bonnefoy, A. Tikan, F. Copie, P. Suret, G. Ducrozet, G. Prabhudesai, G. Michel, A. Cazaubiel, E. Falcon, G. El, and S. Randoux, Phys. Rev. Fluids **5**, 034802 (2020).
[51] A. Osborne, *Nonlinear Ocean Waves* (Academic Press, New York, 2010).
[52] A. Gelash, D. Agafontsev, V. Zakharov, G. El, S. Randoux, and P. Suret, Phys. Rev. Lett. **123**, 234102 (2019).
[53] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.264101 for numerical simulations of the water wave experiments together with a description of the methods used for nonlinear spectral analysis of the wave fields, which includes Refs. [54–60].
[54] S. Randoux, P. Suret, and G. El, Sci. Rep. **6**, 29238 (2016).
[55] A. Goullet and W. Choi, Phys. Fluids **23**, 016601 (2011).
[56] D. G. Dommermuth and D. K. Yue, J. Fluid Mech. **184**, 267 (1987).
[57] B. J. West, K. A. Brueckner, R. S. Janda, D. M. Milder, and R. L. Milton, J. Geophys. Res. **92**, 11803 (1987).
[58] G. Ducrozet, F. Bonnefoy, D. L. Touzé, and P. Ferrant, Eur. J. Mech. B. Fluids **34**, 19 (2012).
[59] Ecole Centrale Nantes, LHEEA, Open-source release of HOS-NWT, https://github.com/LHEEA/HOS-NWT.
[60] F. Bonnefoy, G. Ducrozet, D. L. Touzé, and P. Ferrant, Time domain simulation of nonlinear water waves using spectral methods, in *Advances in Numerical Simulation of Nonlinear Water Waves* (World Scientific, Singapore, 2010), pp. 129–164, https://doi.org/10.1142/9789812836502_0004.
[61] I. S. Chekhovskoy, O. V. Shtyrina, M. P. Fedoruk, S. B. Medvedev, and S. K. Turitsyn, Phys. Rev. Lett. **122**, 153901 (2019).
[62] S. Randoux, P. Suret, A. Chabchoub, B. Kibler, and G. El, Phys. Rev. E **98**, 022219 (2018).

Supplemental Material for "Nonlinear spectral synthesis of soliton gases in deep-water surface gravity waves"

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The purpose of this Supplemental Material is to provide some mathematical, numerical and experimental details that are utilized in the Letter. All equation, figure, reference numbers within this document are prepended with "S" to distinguish them from corresponding numbers in the Letter.

I. INVERSE SCATTERING TRANSFORM ANALYSIS OF THE EXPERIMENTAL DATA

In this Section, we describe briefly the method used to compute the discrete IST spectrum from the signals recorded in the water wave experiment.

The first step for performing the nonlinear analysis of the signals (water elevation given by $\eta(Z,T) = Re\left[A(Z,T)e^{i(k_0 Z-\omega_0 T)}\right]$) recorded in the experiments consists in determining the complex envelope $A(Z,T)$ of the wavefield. This is achieved by using standard techniques based on the Hilbert transform, as discussed e. g. in ref. [1]. Then, physical quantities are put to dimensionless form using the connection between physical and dimensionless variables that are provided in the Letter and recalled here for the sake of simplicity: $\psi = A/\sqrt{\langle|A_0(T)|^2\rangle}$, $T = Z/L_{NL}$, $x = (T - Z/C_g)\sqrt{g/(2L_{NL})}$ with the nonlinear length being defined as $L_{NL} = 1/(\alpha k_0^3\langle|A_0(T)|^2\rangle)$. The brackets denote average over time. Finally the IST discrete spectrum is determined by solving the Zakharov-Shabat system using the Fourier collocation method and following a procedure used and described in ref. [2–5]

II. INTEGRABLE VERSUS NON-INTEGRABLE DYNAMICS IN THE ENSEMBLE OF 16 SOLITONS

In this Section, we use numerical simulations of the focusing 1D-NLSE and of a modified (non-integrable) 1D-NLSE to show the role of higher order effects on the observed space-time dynamics and on the spectral (IST) features that characterize the evolution of the ensemble of 16 solitons considered in Fig. 1 of the Letter.

Following the work reported in ref. [6], higher-order effects in 1D water wave experiments can be described by a modified NLSE written under the form of a spatial evolution equation

$$\frac{\partial A}{\partial Z} = i\frac{k_0}{\omega_0^2}\frac{\partial^2 A}{\partial T^2} + i\alpha k_0^3|A|^2A - \frac{k_0^3}{\omega_0}\left(6|A|^2\frac{\partial A}{\partial T} + 2A\frac{\partial|A|^2}{\partial T} - 2iA\mathcal{H}\left[\frac{\partial|A|^2}{\partial T}\right]\right), \tag{S1}$$

where $A(Z,T)$ represents the complex envelope of the wave field and $\mathcal{H}$ is the Hilbert transform defined by $\mathcal{H}[f] = (1/\pi)\int_{-\infty}^{+\infty} f(\xi)/(\xi - T)d\xi$. When the last three terms are neglected in Eq. (S1), the integrable 1D-NLSE is recovered.

Neglecting the last three terms in Eq. (S1), Fig. S1 shows results obtained from the numerical simulation of Eq. (S1) (integrable focusing 1D-NLSE) for the ensemble of 16 solitons considered in the experiments reported in Fig. 1 of the Letter. Fig. S1(a) (resp. Fig. S1(b)) shows the modulus $|A(Z,T)|$ of the wavefield that is computed at $Z_1 = 6$ m (resp. at $Z_{20} = 120$ m). Despite the significantly complicated space time evolution shown in Fig. S1(e) over the 120 m-long propagation distance, the dynamics is integrable which implies that the discrete IST spectrum remains perfectly unchanged between Z_1 and Z_{20}, compare Fig. S1(c) and Fig.S1(d).

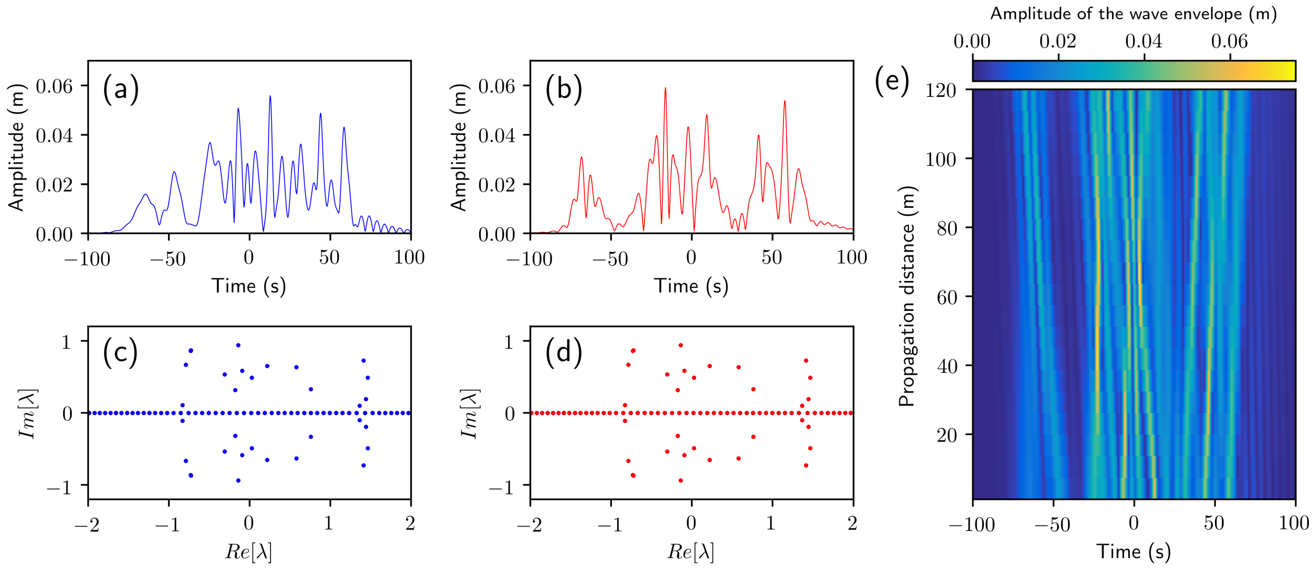

FIG. S1: Integrable dynamics. Numerical simulations of the integrable focusing 1D-NLSE (Eq. (S1) where the last three terms are neglected) for the ensemble of 16 solitons considered in Fig. 1 of the Letter. (a) Modulus $|A(Z_1, T)|$ of the wave envelope at $Z_1 = 6$ m and (c) corresponding discrete IST spectrum (red points). (b) Modulus $|A(Z_{20}, t)|$ of the wave envelope at $Z_{20} = 120$ m and (d) corresponding discrete IST spectrum. (e) Space-time plot showing the nonlinear evolution of the modulus $|A(Z, T)|$ of the wave envelope.

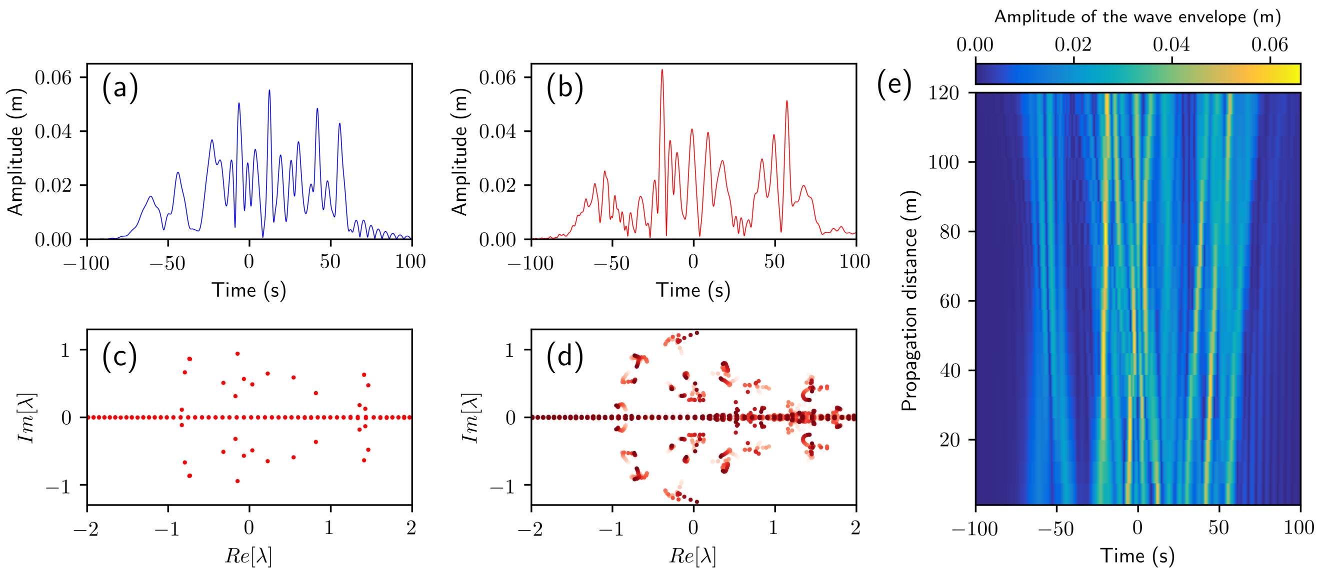

FIG. S2: Non-integrable dynamics. Numerical simulations of Eq. (S1) for the ensemble of 16 solitons considered in Fig. 1 of the Letter. (a) Modulus $|A(Z_1, t)|$ of the wave envelope at $Z_1 = 6$ m and (c) corresponding discrete IST spectrum. (b) Modulus $|A(Z_{20}, t)|$ of the wave envelope at $Z_{20} = 120$ m . (d) Space evolution of the discrete IST spectra along the tank from $Z_1 = 6$ m (light red) to $Z_{20} = 120$ m (dark red). (e) Space-time plot showing the nonlinear evolution of the modulus $|A(Z, T)|$ of the wave envelope.

If higher order effects described by the three last terms in Eq. (S1) are taken into account, the space-time evolution is slightly perturbed compared to the integrable case, compare Fig. S2(b) with Fig. S1(b) and Fig. S2(e) with Fig. S1(e). Contrary to results reported in Fig. S1, the isospectrality condition is now not verified because of the higher-order effects that break the integrability of the wave dynamics. Fig. S2(d) shows clearly that each of the 16 discrete eigenvalues composing the random wavefield does not remain invariant over propagation distance but follows an individual trajectory in the complex plane, as already e.g. evidenced in ref. [7] in numerical simulations of a laser system. Similar spectral (IST) results have been presented in experimental results reported in Fig. 1 of the Letter. However clean trajectories in the complex IST plane cannot be clearly identified in experiments because of small calibration errors in the measurement of the wave elevation.

Assuming that the dynamics is governed by Eq. (S1), the length scale L_{HOE} over which higher-order effects perturbe the evolution of the wavefield is given by $L_{HOE} = L_{NL}\Delta\omega/\omega_0$ where $L_{NL} = 1/(\alpha k_0^3 \langle |A_0(T)|^2 \rangle)$ and $\Delta\omega$ is the typical spectral bandwidth of the wavefield.

III. DIRECT NUMERICAL SIMULATIONS OF EULER'S EQUATIONS FOR THE GAS OF 128 SOLITONS

Direct numerical simulations of the Euler's equations have been performed with the efficient and accurate High-Order Spectral (HOS) method [8, 9]. The numerical model used in our numerical simulations reproduce the main features of the water tank, namely: i) the generation of waves through a wave maker and ii) the absorption of reflected waves with an absorbing beach. To this end a Numerical Wave Tank, entitled HOS-NWT, has been developed [10] (the code being available open-source [11]). It uses the exact same wave makers motions than in the experiments for a simplified comparison procedure. This specific model has been widely validated in different configurations and more details can be found in [10, 12].

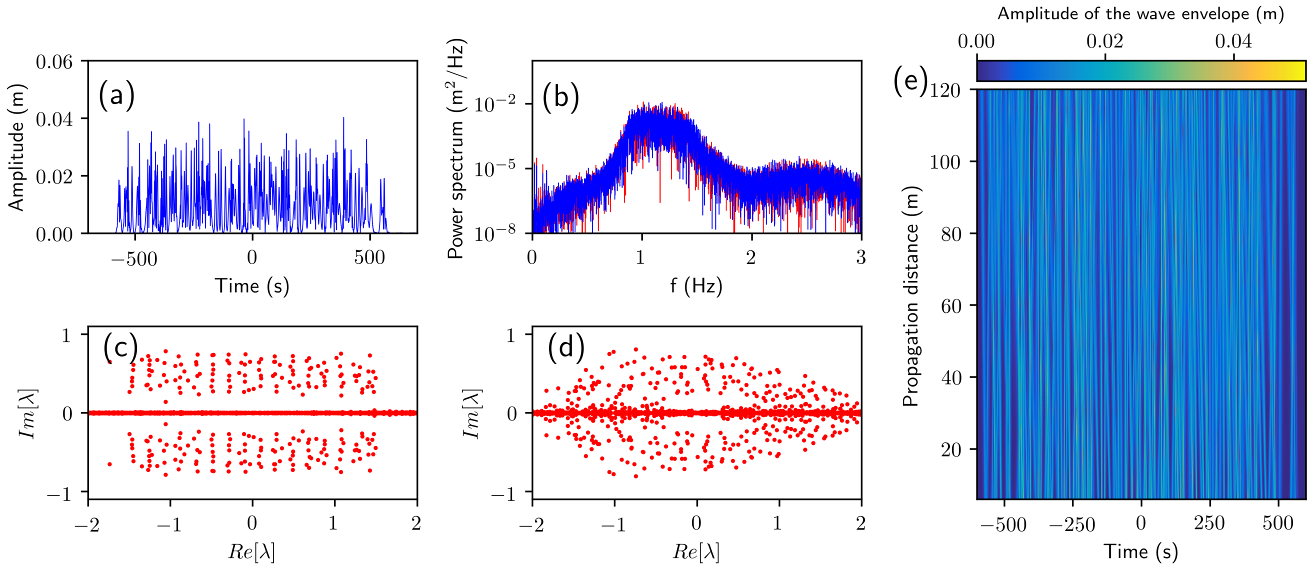

FIG. S3: Numerical simulations of the Euler's equations corresponding to the experimental results shown in Fig. 2 of the manuscript ($N = 128$). (a) Modulus of the wave envelope at $Z_1 = 6$ m, close to the wavemaker. (b) Fourier power spectra of wave elevation at $Z_1 = 6$ m (blue line) and at $Z_{20} = 120$ m (red line). (c) Discrete IST spectrum measured at $Z_1 = 6$ m. (d) Discrete IST spectrum measured at $Z_{20} = 120$ m. (e) Space-time evolution of modulus of the wave envelope. Parameters of the simulation are $f_0 = 1.15$ Hz, $k_0 = 5.32$ m^{-1}, $L_{NL} = 45$ m ($< |A_0(T)|^2 >= 1.58 \times 10^{-4}$ m^2.

Fig. S3 shows numerical simulations of Euler's equations that are made with physical values characterizing the experiments presented in the Letter for the gas of 128 solitons. The dynamical and spectral features found in numerical simulations reported in Fig. S3 are very similar to those reported in Fig. 2 of the Letter.

[S1] A. Osborne, Nonlinear ocean waves (Academic Press, 2010).
[S2] J. Yang, Nonlinear Waves in Integrable and Non-integrable Systems, Mathematical Modeling and Computation (Society for Industrial and Applied Mathematics, 2010).
[S3] S. Randoux, P. Suret, and G. El, Scientific reports **6**, 29238 (2016).

[S4] S. Randoux, P. Suret, A. Chabchoub, B. Kibler, and G. El, Phys. Rev. E **98**, 022219 (2018).
[S5] F. Bonnefoy, A. Tikan, F. Copie, P. Suret, G. Ducrozet, G. Prabhudesai, G. Michel, A. Cazaubiel, E. Falcon, G. El, et al., Phys. Rev. Fluids **5**, 034802 (2020).
[S6] A. Goullet and W. Choi, Physics of Fluids **23**, 016601 (2011).
[S7] I. S. Chekhovskoy, O. V. Shtyrina, M. P. Fedoruk, S. B. Medvedev, and S. K. Turitsyn, Phys. Rev. Lett. **122**, 153901 (2019).
[S8] D. G. Dommermuth and D. K. Yue, J. Fluid Mech. **184**, 267 (1987).
[S9] B. J. West, K. A. Brueckner, R. S. Janda, D. M. Milder, and R. L. Milton, J. Geophys. Res. **92**, 11803 (1987).
[S10] G. Ducrozet, F. Bonnefoy, D. L. Touz, and P. Ferrant, Eur. J. Mech. B. Fluids **34**, 19 (2012).
[S11] Ecole Centrale Nantes, LHEEA, Open-source release of HOS-NWT, https://github.com/LHEEA/HOS-NWT.
[S12] F. Bonnefoy, G. Ducrozet, D. L. Touz, and P. Ferrant, Time domain simulation of nonlinear water waves using spectral methods (2010), pp. 129–164.