Energy Cascades in MHD

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The MHD equations

$$\partial_t z^{\pm} + z^{\mp}\nabla z^{\pm} = \pm B_0 \nabla z^{\pm} - \nabla P + \eta \nabla^2 z^{\pm} + F^{\pm}$$

$$\nabla \cdot z^{\pm} = 0 \qquad (\eta = \nu)$$

- Conserved energies

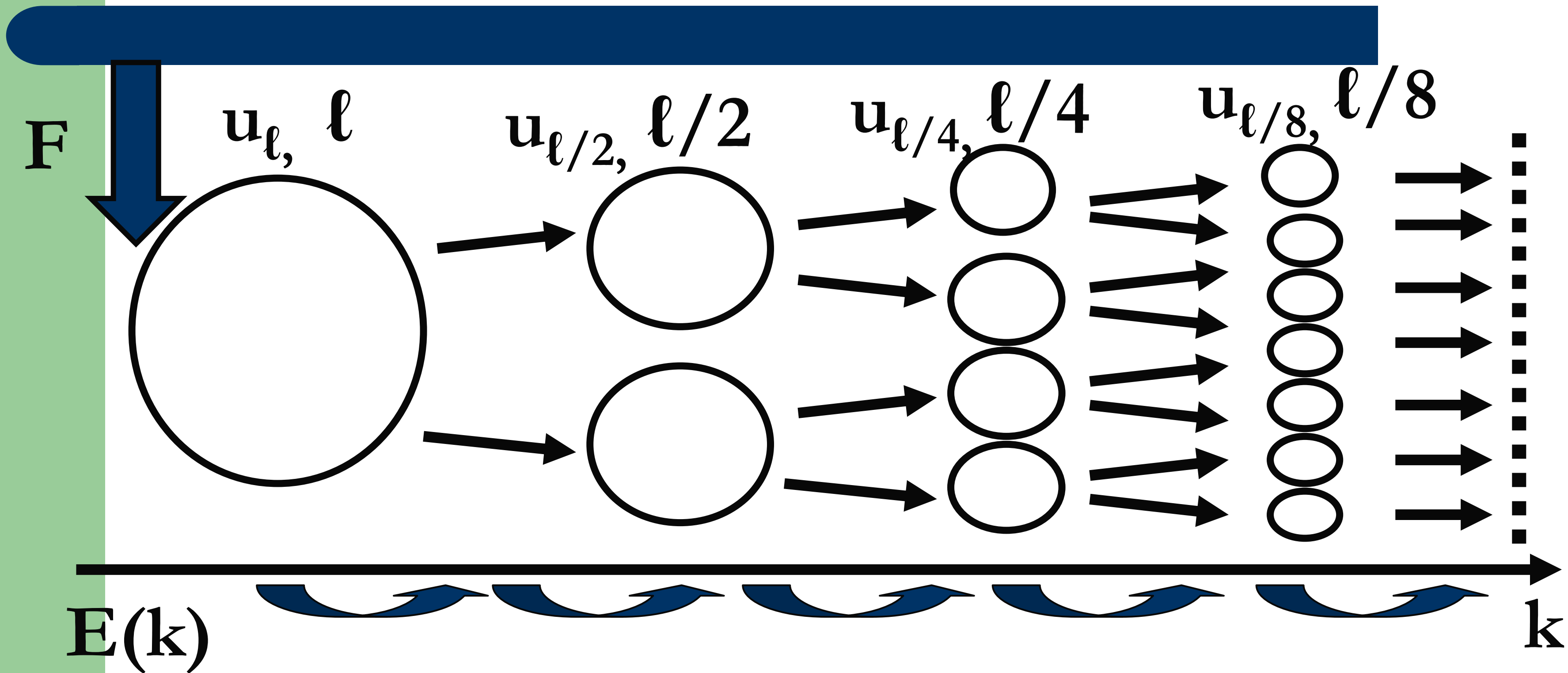
$$\tfrac{1}{2}\partial_t \int \left(z^{\pm}\right)^2 dx^3 = -\eta \left\| \nabla z^{\pm} \right\|^2$$

$$E^{\pm} = \tfrac{1}{2}\int \left(z^{\pm}\right)^2 dx^3$$

Basic questions

- How energy cascades?
- At what rate energy cascades?
- At what direction?
- What is the spectrum?

The isotropic turbulent cascade



$\tau \sim \ell / u_\ell$

$\epsilon \sim u_\ell^2/\tau \sim u_\ell^3/\ell$

$dE/dk \sim E/k \sim u_\ell^2 \cdot \ell \sim k^{-5/3}$

Weak turbulence theory, Galtier (2000)

$$(Z^+{}_k \rightleftarrows Z^-{}_q) \Rightarrow Z^+{}_p$$

$$\left\{\begin{array}{l} k+q+p=0 \\ \omega_k+\omega_q+\omega_p=0, \quad Bk_z-Bq_z+Bp_z=0 \end{array}\right\}$$

The wave resonance conditions

$$q_z=0$$

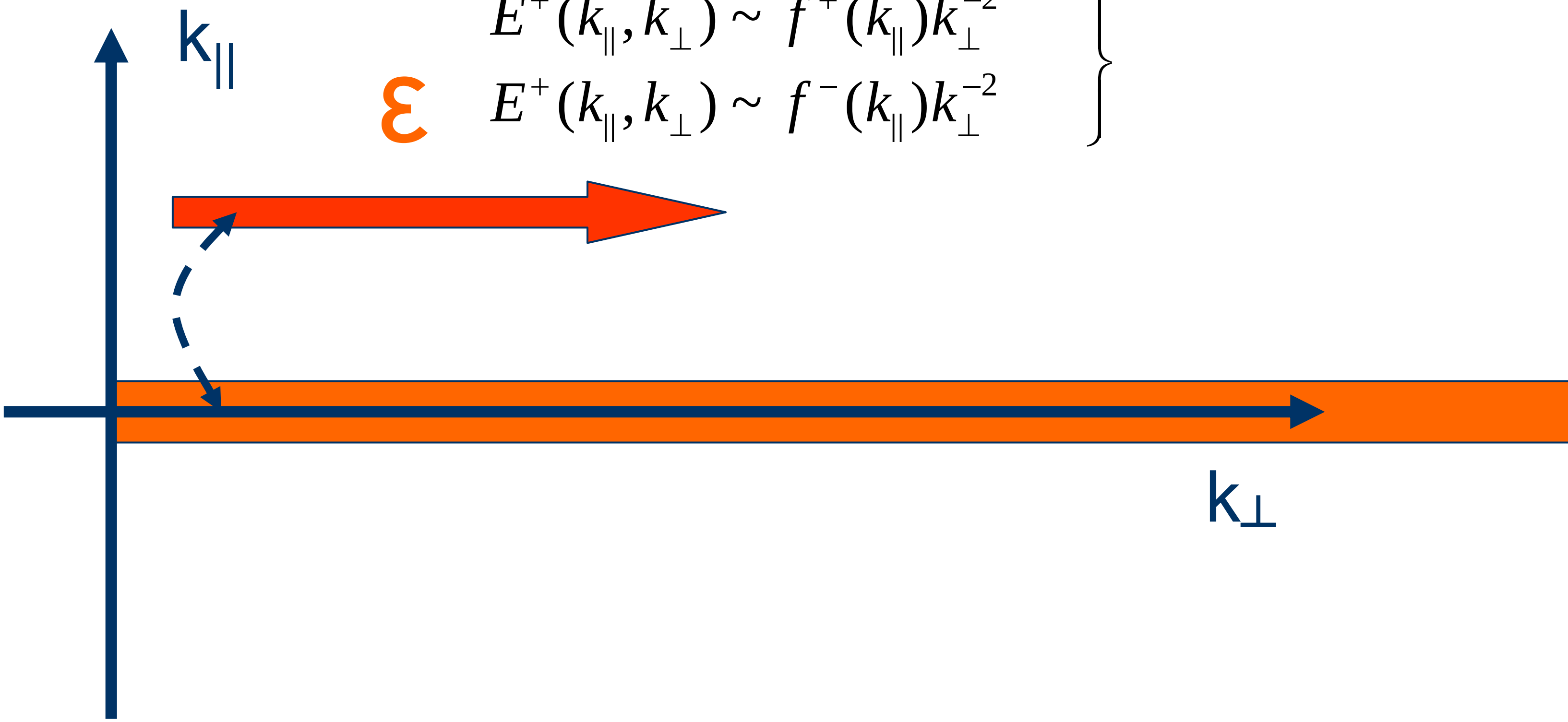
$$\left.\begin{array}{ll} E^+(k_{\|},k_{\perp}) \sim f^+(k_{\|})k_{\perp}^{-n^+} & E^+(0,k_{\perp}) \sim k_{\perp}^{-m^+} \\ E^+(k_{\|},k_{\perp}) \sim f^-(k_{\|})k_{\perp}^{-n^-} & E^-(0,k_{\perp}) \sim k_{\perp}^{-m^-} \end{array}\right\}$$

$$m^{\pm}+n^{\mp}=4$$

If smooth $E(k_{\|},k_{\perp})$ across $k_{\|}=0$ $\qquad m^{\pm}=n^{\mp}=2$

Weak turbulence theory, Galtier (2000)

$$
\left.
\begin{aligned}
E^{+}(k_{\|}, k_{\perp}) &\sim f^{+}(k_{\|}) k_{\perp}^{-2} \\
E^{+}(k_{\|}, k_{\perp}) &\sim f^{-}(k_{\|}) k_{\perp}^{-2}
\end{aligned}
\right\}
$$



Models in MHD

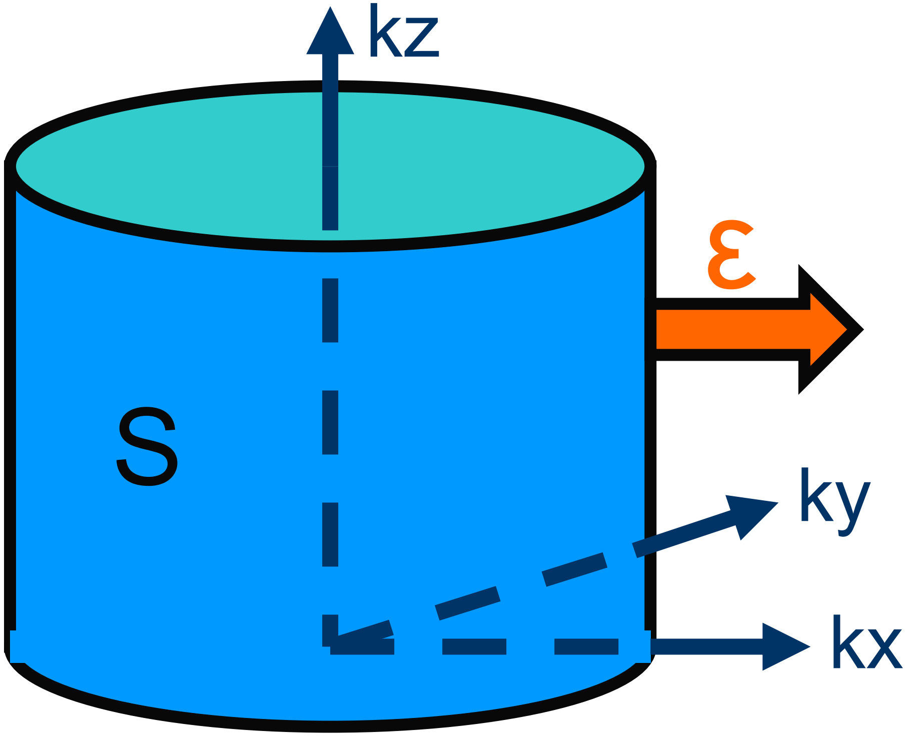
- Iroshnikov-Kraichnan $E \sim k^{-3/2}$
- Goldreich-Shridhar $E \sim k_{\perp}^{-5/3}$, $k_{\parallel} \sim k_{\perp}^{2/3}$
- Bhattacharjee-Ng $E \sim k_{\perp}^{-2} k_{\parallel}^{-1}$
- Zhou-Matthaus-Dmitruk $T^{-1} = T_{A}^{-1} + T_{NL}^{-1}$
- Galtier et al. $E \sim k_{\perp}^{-5/3}$, $k_{\parallel} \sim \chi k_{\perp}^{2/3}$
- Boldyrev $E \sim k_{\perp}^{-3/2}$, $k_{\parallel} \sim k_{\perp}^{2/3}$

All assume local interactions

Measuring Fluxes

- Energy Flux

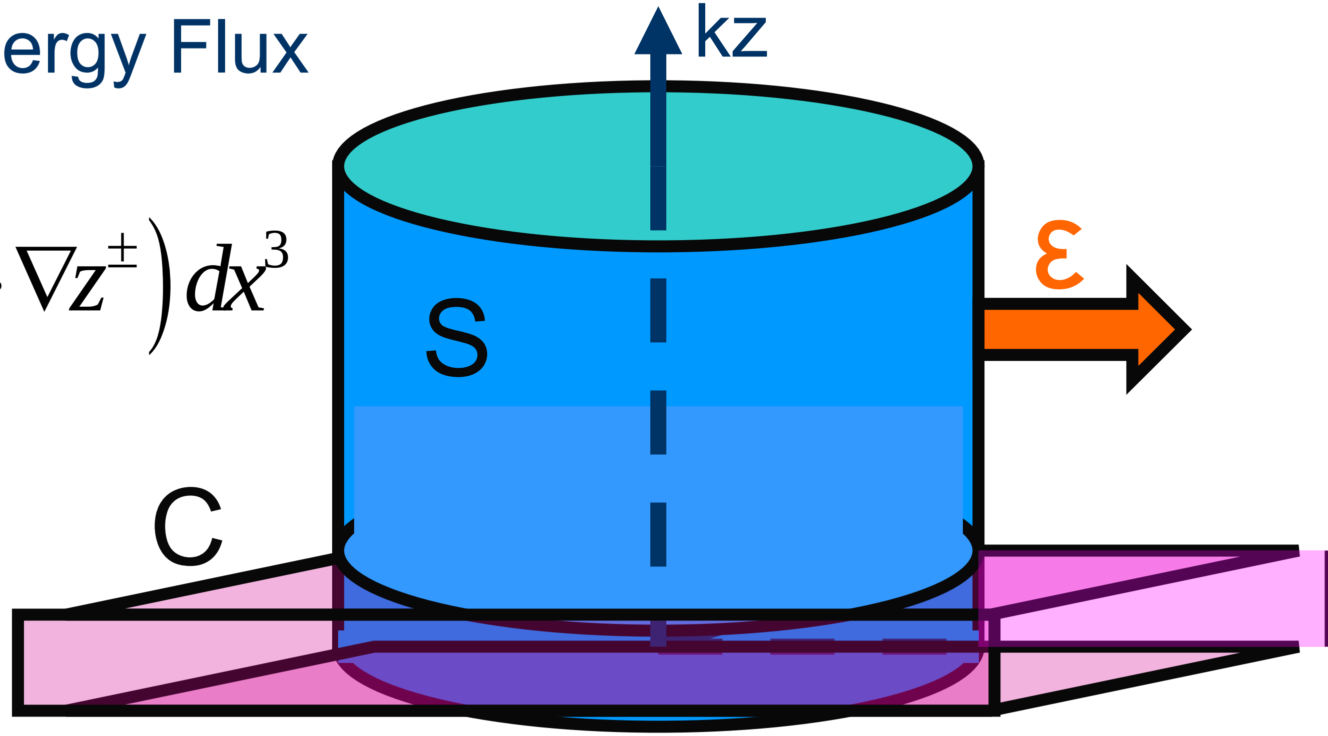
$$\Pi^{\pm}(S) = \int z_S^{\pm}\left(z^{\mp} \cdot \nabla z^{\pm}\right) dx^3$$



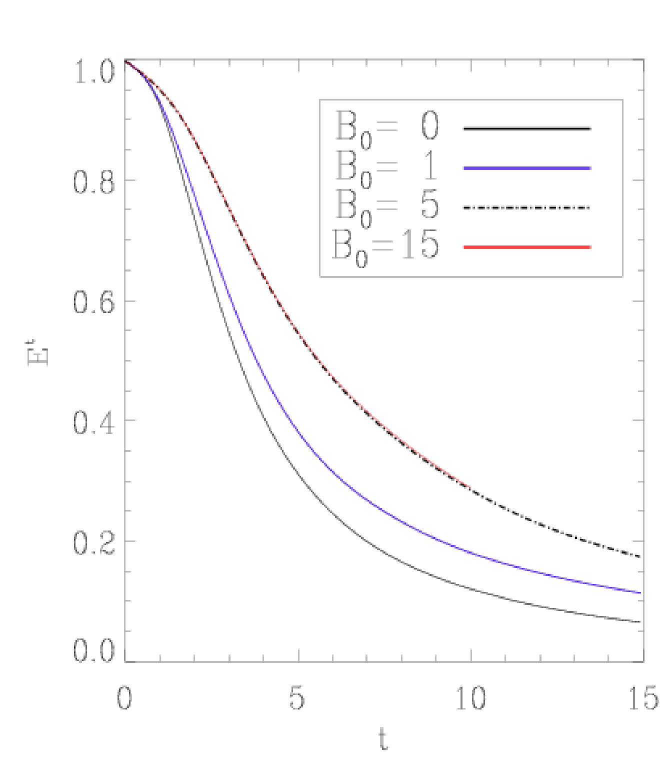
Partial Fluxes

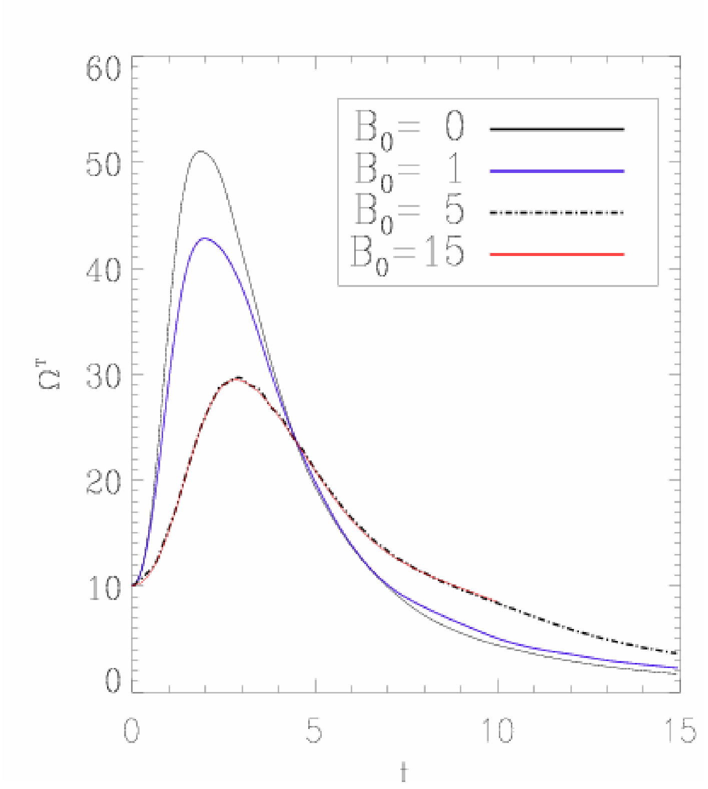
- Partial Energy Flux

$$\Pi_C^{\pm}(S) = \int z_S^{\pm}\left(z_C^{\mp}\cdot\nabla z^{\pm}\right)dx^3$$



A decaying MHD run. B_0=0, B_0=1, B_0=5 and B_0=15 (Bigot et al)





Partial energy Fluxes I ($k_\perp$ direction)

- B=0
- B=1
- B=5
- B_0=15

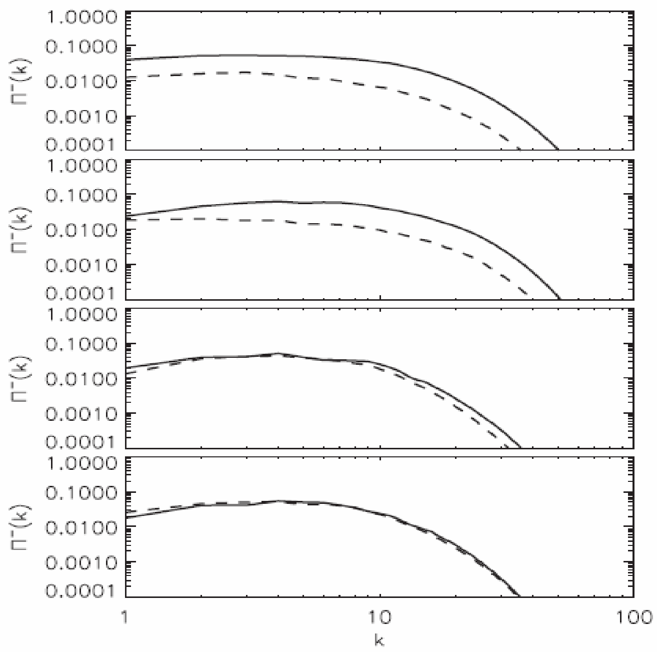


FIG. 8. Total energy flux $\Pi^{-}(K)$ (solid line) across cylinders of radius K together with the partial flux $\Pi^{-}_{P=0}(K)$ (dashed line) for the four different values of B, from $B=0$ (top panel) up to $B=15$ (bottom panel).

Total Flux

Flux due to the k_z=0 modes

Partial energy Fluxes II ($k_{||}$ direction)

- B=0
- B=1
- B=5
- B_0=15

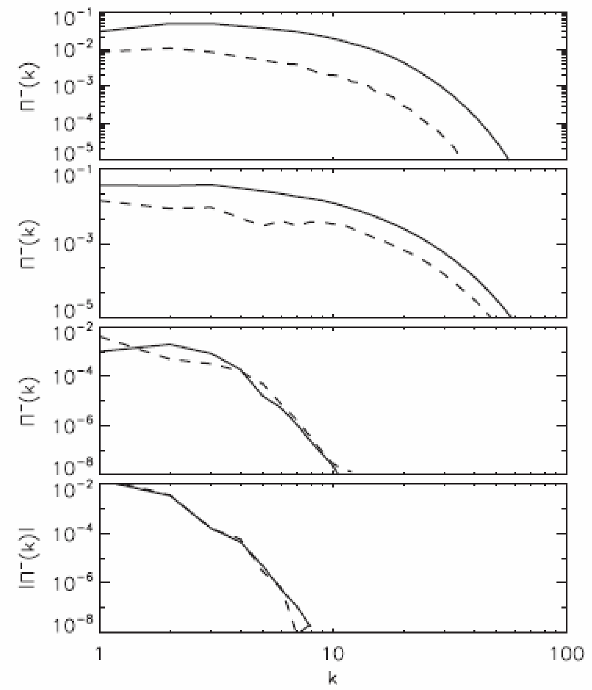
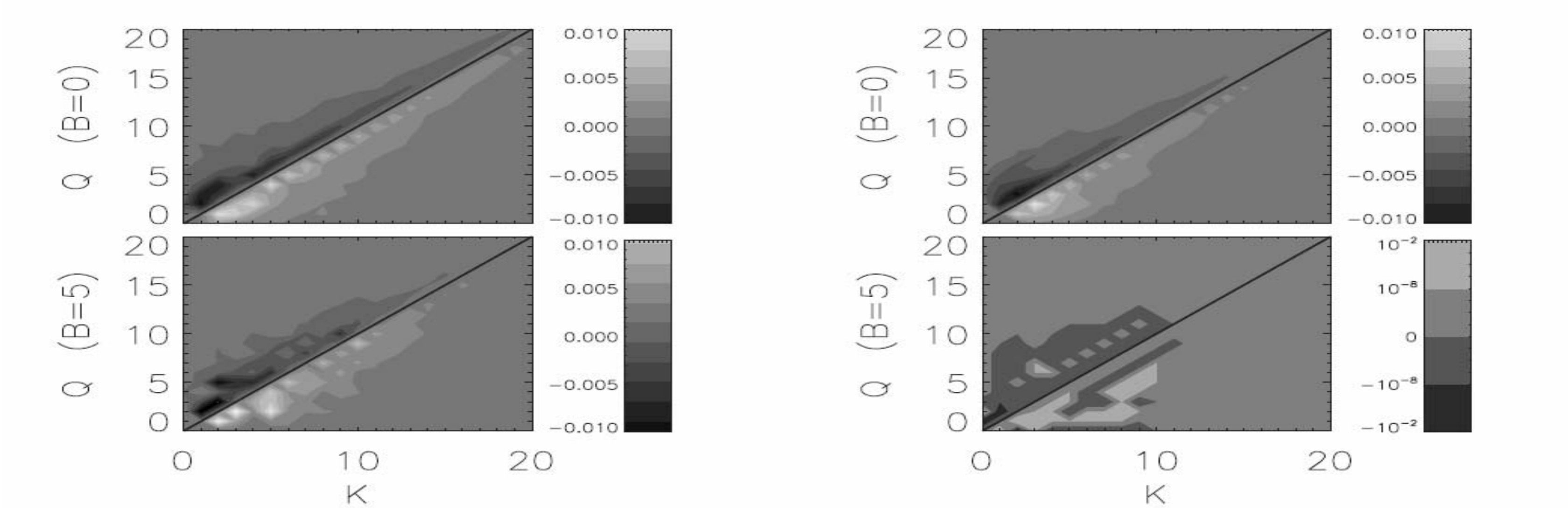


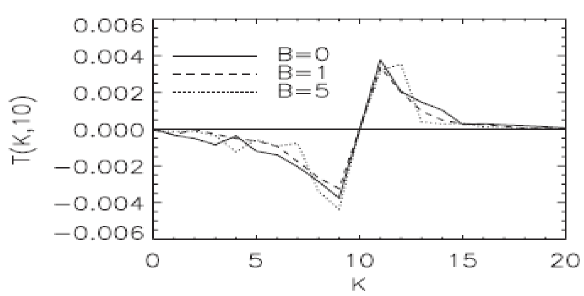
FIG. 9. Total energy flux $\Pi^{-}(K)$ (solid line) across planes at $k_{\parallel} = K$ together with the partial flux $\Pi^{-}_{P=1}(K)$ (dashed line) for the four different values of B from $B=0$ (top panel) up to $B=15$ (bottom panel). Note the absolute value in the latter case.

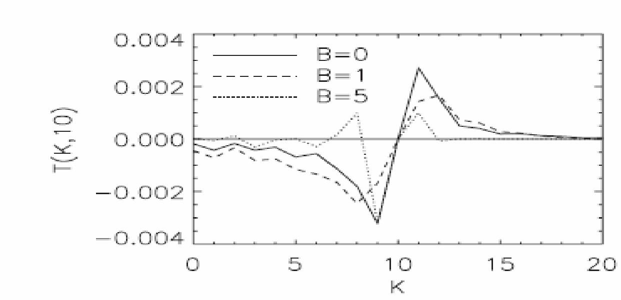
Total Flux

Flux due to the k_z=0 modes

Locality of transfer

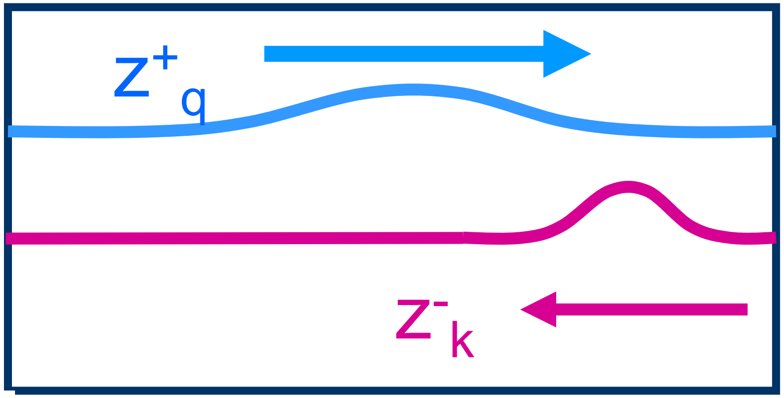






A simple model

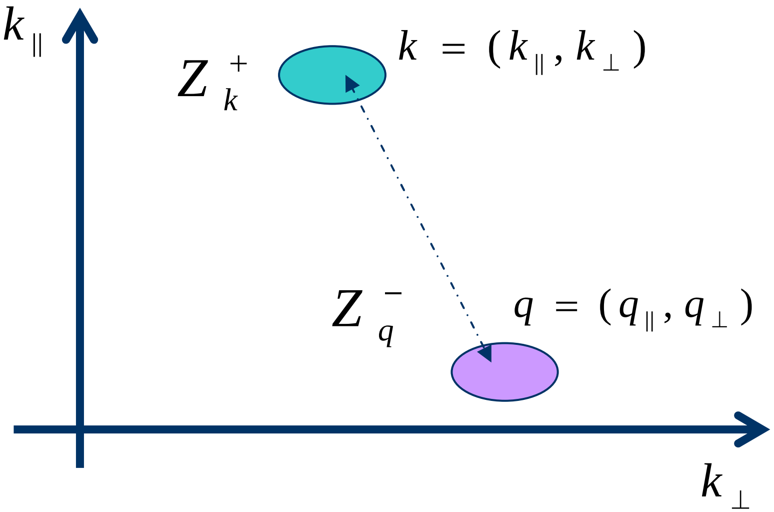
Consider the cascade of $E^{\pm}$ due to interactions of different length scale eddies Z_k and Z_q with $k \neq q$



Where $E^{\pm}(k_{\parallel}, k_{\perp})$ the Energy spectrum such that

$$E^{\pm} = \int E(k_{\perp}, k_{\parallel}) k_{\perp} dk_{\perp} dk_{\parallel}$$

$$Z_k^{\ 2} \sim k_{\perp} k_{\parallel} E(k_{\perp}, k_{\parallel})$$



Weak turbulence limit

- Consider only cascade in the $\perp$ direction

$$\varepsilon^{+}{}_{\perp} = (Z_k^{+})^2 Z_q^{-} / \ell_{q_\perp} = \quad k_{||} k_{\perp} E(k_{||}, k_{\perp}) \; q_{\perp} \sqrt{q_{||} q_{\perp} E(q_{||}, q_{\perp})}$$

- Allow only wave numbers that satisfy the resonance condition

$$\tau_A \sim \tau_{NL} \Rightarrow \quad B q_{||} \sim Z_q^{-} q_{\perp}, \qquad q_{\perp} = k_{\perp}$$

- or $q_{||} = k_{\perp}^3 E(q_{||}, q_{\perp}) / B^2 << 1$

$$\varepsilon_{\perp} = k_{||} k_{\perp}{}^4 E^{+}(k_{||}, k_{\perp}) \; E^{-}(0, k_{\perp}) / B$$

Weak turbulence limit

$$\varepsilon_\perp = k_{||} k_\perp{}^4 E^+(k_{||}, k_\perp)\ E^-(0, q_\perp) / B$$

$$\left.\begin{array}{ll} E^+(k_{||}, k_\perp) \sim f^+(k_{||}) k_\perp^{-n^+} & E^+(0, k_\perp) \sim k_\perp^{-m^+} \\ E^+(k_{||}, k_\perp) \sim f^-(k_{||}) k_\perp^{-n^-} & E^-(0, k_\perp) \sim k_\perp^{-m^-} \end{array}\right\} \quad m^\pm + n^\mp = 4$$

If smooth $E(k_{||},k_\perp)$ across $k_{||}=0$ $\qquad m^\pm = n^\mp = 2$

Stronger turbulence

$$\partial_t E + \nabla \cdot \vec{\varepsilon} = 0$$

$$\varepsilon^+{}_\perp = (Z_k^+)^2 Z_q^- / \ell_{q_\perp} = \quad k_{||} k_\perp E(k_{||}, k_\perp)\; q_\perp \sqrt{q_{||} q_\perp E(q_{||}, q_\perp)}$$

$$\varepsilon^+{}_{||} = (Z_k^+)^2 Z_q^- / \ell_{q_{||}} = \quad k_{||} k_\perp E(k_{||}, k_\perp)\; q_{||} \sqrt{q_{||} q_\perp E(q_{||}, q_\perp)}$$

$$\tau_A \sim \tau_{NL} \Rightarrow \quad B q_{||} \sim Z_q^- q_\perp, \qquad q_\perp = k_\perp$$

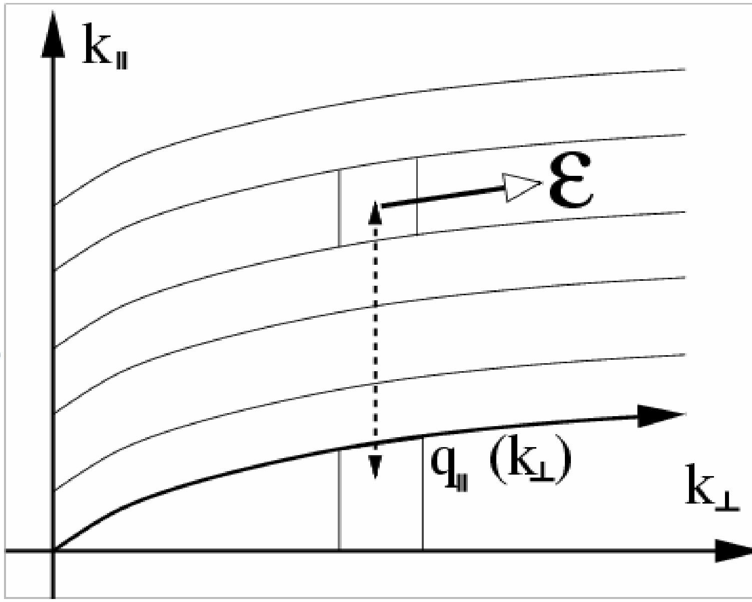
Stronger turbulence

- "Stronger" turbulence theory spectrum

$$E(k_\perp, k_\parallel) \simeq \varepsilon^{2/3} k_\perp^{-5/3} k_\parallel^{-1}$$

- Energy cascades along the Lines $k_\parallel = C + k_\perp^{2/3} \varepsilon^{1/3} / B$

- Interactions with the manifold: $q_\parallel = k_\perp^{2/3} \varepsilon^{1/3} / B$



Conclusions

- Non-locality is essential for MHD in the presence of guiding field
- Flux is Anisotropic (not just the spectrum)
- A simple assumption nonlocal model can be constructed