Dynamical Solutions of the 3-Wave Kinetic Equations

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Outline

3-Wave Turbulence

$$H = T + U = \int \omega_{\mathbf{k}} a_{\mathbf{k}} \bar{a}_{\mathbf{k}} d\mathbf{k} + \int u(\mathbf{k}) d\mathbf{k}$$

Forcing and dissipation are added to Hamilton's equations:

$$\frac{\partial a_{\mathbf{k}}}{\partial t} = i \frac{\delta H}{\delta \bar{a}_{\mathbf{k}}} + f_{\mathbf{k}} - \gamma_{\mathbf{k}} a_{\mathbf{k}}$$

$a_{\mathbf{k}}$, $\bar{a}_{\mathbf{k}}$ are complex canonical variables. Interaction energy:

$$u(\mathbf{k}_1) = \int V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \left(a_{\mathbf{k}_1} a_{\mathbf{k}_2} \bar{a}_{\mathbf{k}_3} + \bar{a}_{\mathbf{k}_1} \bar{a}_{\mathbf{k}_2} a_{\mathbf{k}_3} \right) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\, d\mathbf{k}_2 d\mathbf{k}_3$$

Scaling parameters :	Dimension, d:	$\mathbf{k} \in \mathbf{R}^d$
(d, α, γ)	Dispersion, α:	$\omega_{\mathbf{k}} \sim k^{\alpha}$
	Nonlinearity, γ:	$V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \sim k^{\gamma}$

The 3-wave kinetic equation

Evolution of WT wave spectrum, $n_{\mathbf{k}}$, given by:

$$\begin{aligned}
\frac{\partial n_{\mathbf{k}_1}}{\partial t} &= \pi \int V^2_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\left(a_1 n_{\mathbf{k}_2} n_{\mathbf{k}_3} - a_2 n_{\mathbf{k}_1} n_{\mathbf{k}_2} - a_3 n_{\mathbf{k}_1} n_{\mathbf{k}_3}\right) \\
&\qquad \delta(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3})\,\delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\,d\mathbf{k}_2 d\mathbf{k}_3 \\
&+\pi \int V^2_{\mathbf{k}_2\mathbf{k}_1\mathbf{k}_3}\left(a_1 n_{\mathbf{k}_2} n_{\mathbf{k}_3} + a_2 n_{\mathbf{k}_1} n_{\mathbf{k}_2} - a_3 n_{\mathbf{k}_1} n_{\mathbf{k}_3}\right) \\
&\qquad \delta(\omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1})\,\delta(\mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_1)\,d\mathbf{k}_2 d\mathbf{k}_3 \\
&+\pi \int V^2_{\mathbf{k}_3\mathbf{k}_1\mathbf{k}_2}\left(a_1 n_{\mathbf{k}_2} n_{\mathbf{k}_3} - a_2 n_{\mathbf{k}_1} n_{\mathbf{k}_2} + a_3 n_{\mathbf{k}_1} n_{\mathbf{k}_3}\right) \\
&\qquad \delta(\omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})\,\delta(\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\,d\mathbf{k}_2 d\mathbf{k}_3 \\
&= a_1 S_1[n_{\mathbf{k}}] + a_2 S_2[n_{\mathbf{k}}] + a_3 S_3[n_{\mathbf{k}}]
\end{aligned}$$

$a_1 = a_2 = a_3 = 1$!

Isotropic Kinetic Equation: Forward Transfer

Angle averaged spectrum, $N_\omega = \frac{\Omega_d}{\alpha}\omega^{\frac{d-\alpha}{\alpha}} n_\omega$, satisfies:

$$\frac{\partial N_{\omega_1}}{\partial t} = a_1 S_1[N_\omega] + a_2 S_2[N_\omega] + a_3 S_3[N_\omega]$$

$$\begin{aligned} S_1[N_{\omega_1}] = & \int L_1(\omega_2,\omega_3)\, N_{\omega_2}\, N_{\omega_3}\, \delta(\omega_1-\omega_2-\omega_3)\, d\omega_2 d\omega_3 \\ & - \int L_1(\omega_3,\omega_1)\, N_{\omega_3}\, N_{\omega_1}\, \delta(\omega_2-\omega_3-\omega_1)\, d\omega_2 d\omega_3 \\ & - \int L_1(\omega_1,\omega_2)\, N_{\omega_1}\, N_{\omega_2}\, \delta(\omega_3-\omega_1-\omega_2)\, d\omega_2 d\omega_3, \end{aligned}$$

Details are hidden in the kernel $L(\omega_1,\omega_2)$. Scaling of the interaction coefficient:

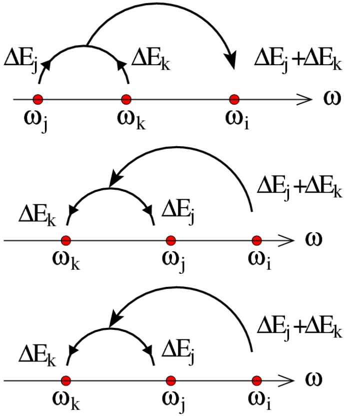
$$L_1(\omega_1,\omega_2) \sim \omega^\lambda, \qquad \lambda = \frac{2\gamma-\alpha}{\alpha}$$

Isotropic Kinetic Equation: Backward Transfer

$$
\begin{aligned}
S_2[N_{\omega_1}] =& -\int L_2(\omega_2,\omega_3)\, N_{\omega_1}\, N_{\omega_2}\, \delta(\omega_1-\omega_2-\omega_3)\, d\omega_2 d\omega_3 \\
&+\int L_2(\omega_3,\omega_1)\, N_{\omega_2}\, N_{\omega_3}\, \delta(\omega_2-\omega_3-\omega_1)\, d\omega_2 d\omega_3 \\
&+\int L_2(\omega_1,\omega_2)\, N_{\omega_3}\, N_{\omega_1}\, \delta(\omega_3-\omega_1-\omega_2)\, d\omega_2 d\omega_3,
\end{aligned}
$$

$$
\begin{aligned}
S_3[N_{\omega_1}] =& -\int L_3(\omega_2,\omega_3)\, N_{\omega_1}\, N_{\omega_2}\, \delta(\omega_1-\omega_2-\omega_3)\, d\omega_2 d\omega_3 \\
&+\int L_3(\omega_3,\omega_1)\, N_{\omega_2}\, N_{\omega_3}\, \delta(\omega_2-\omega_3-\omega_1)\, d\omega_2 d\omega_3 \\
&+\int L_3(\omega_1,\omega_2)\, N_{\omega_3}\, N_{\omega_1}\, \delta(\omega_3-\omega_1-\omega_2)\, d\omega_2 d\omega_3.
\end{aligned}
$$

Physical Meaning of the $S_i[N_\omega]$: Triad Formulation



Rates:

- $S_1[N_\omega]$
  Loss ω_j: $\omega_j\, L_1(\omega_j, \omega_k)\, N_{\omega_j}\, N_{\omega_k}$.
  Loss ω_k: $\omega_k\, L_1(\omega_j, \omega_k)\, N_{\omega_j}\, N_{\omega_k}$.
- $S_2[N_\omega]$
  Gain ω_j: $\omega_j\, L_2(\omega_j, \omega_k)\, N_{\omega_i}\, N_{\omega_j}$.
  Gain ω_k: $\omega_k\, L_2(\omega_j, \omega_k)\, N_{\omega_i}\, N_{\omega_j}$.
- $S_3[N_\omega]$
  Loss ω_j: $\omega_j\, L_3(\omega_j, \omega_k)\, N_{\omega_i}\, N_{\omega_k}$.
  Loss ω_k: $\omega_k\, L_3(\omega_j, \omega_k)\, N_{\omega_i}\, N_{\omega_k}$.

The Kolmogorov-Zakharov Spectrum

Zakharov transformation yields stationary solution:

$$N_\omega = c_{\mathrm{KZ}} \sqrt{J}\, \omega^{-\frac{\lambda+3}{2}}.$$

where

$$c_{\mathrm{KZ}} = \sqrt{\frac{2}{A}}, \qquad A = \left.\frac{dI}{dx}\right|_{x=\frac{\lambda+3}{2}}.$$

and

$$I(x) = \frac{1}{2}\int_0^1 L_1(y,1-y)\,(y(1-y))^{-x}\,(1-y^x-(1-y)^x)\,(1-y^{2x-\lambda-2}-(1-y)^{2x-\lambda-2})\,dy.$$

Numerical Solution of the Isotropic Kinetic Equation

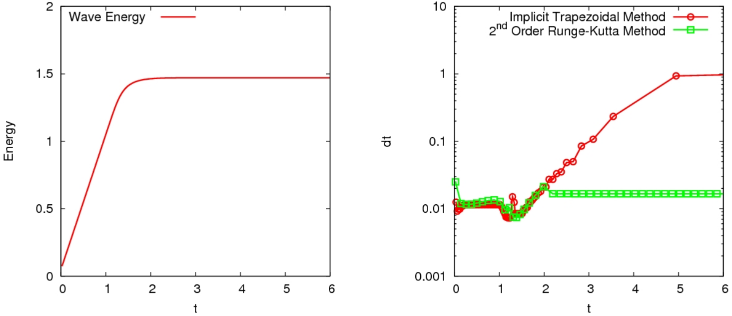
For various reasons one may be interested in more than just the KZ solution. There are no known exact solutions.
Discrete case: $\mathbf{N} = (N_1, N_2, N_3, \ldots)$. $N_i = N(\omega_i)$, $\omega_i = i\Delta\,\omega$.
Reduces to a large set of coupled ODEs for $\mathbf{N}$:

$$\frac{d\mathbf{N}}{dt} = S[\mathbf{N}] = S_1[\mathbf{N}] + S_2[\mathbf{N}] + S_3[\mathbf{N}]$$

Numerical solution presents some particular difficulties:

- Widely varying timescales $\Rightarrow$ use adaptive timestepping.
- System is very stiff $\Rightarrow$ require *implicit solver*.
- Need to resolve very many modes to measure scaling exponents $\Rightarrow$ need to approximate the collision integrals.

Stiffness: $L(\omega_1, \omega_2) = \omega_1^2 + \omega_2^2$, 1000 modes



Implicit trapezoidal rule (stepwise error of h^3):

$$\mathbf{N}(t+h) - \mathbf{N}(t) - \frac{1}{2}\, h\, [S[\mathbf{N}(t)] + S[\mathbf{N}(t+h)]]$$

Solved via Rosenbrock algorithm.

Computing the Collision Integrals 1

- Divide frequency domain into bins $B_i = [\omega_i^{(L)}, \omega_i^{(R)}]$ having characteristic frequencies $\widetilde{\Omega}_i = \frac{1}{2}(\omega_i^{(R)} + \omega_i^{(L)})$ exponentially spaced (except for the first few).
- Apply triad formulation of collision integrals to compute *effective* energy transfer between bins rather than between individual modes.
- $S_1[\mathbf{N}]$ requires us to approximate integrals of the form

$$\int_{\omega_i^{(L)}}^{\omega_i^{(R)}} d\,\omega_i \int_{\omega_j^{(L)}}^{\omega_j^{(R)}} d\,\omega_j \;(\omega_i + \omega_j)\, L(\omega_i, \omega_j)\, N(\omega_i)\, N(\omega_j)$$

- Approximation: if $\widetilde{\Omega}_j <= \widetilde{\Omega}_i$ treat all waves in B_j as having frequency $\widetilde{\Omega}_j$. (H. Lee 2001)

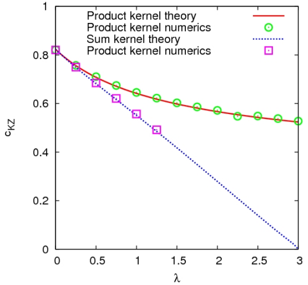
Computing the Collision Integrals 2

- Thus we obtain *one-dimensional* integrals which can be done by quadrature:

$$
\begin{aligned}
\Delta E_j &= \widetilde{\Omega}_j \, N(\widetilde{\Omega}_j) \int_{\omega_i^{(L)}}^{\omega_i^{(R)}} d\,\omega_i, L(\omega_i, \omega_j)\, N(\omega_i) \\
\Delta E_i &= N(\widetilde{\Omega}_j) \int_{\omega_i^{(L)}}^{\omega_i^{(R)}} d\,\omega_i, L(\omega_i, \omega_j)\, \omega_i \, N(\omega_i) \\
\Delta E_k &= \Delta E_i + \Delta E_j.
\end{aligned}
$$

- Similar expressions for $S_2[N_\omega]$ and $S_3[N_\omega]$.
- Many technical details not worth discussing.

Validation of the algorithm



Computing c_{KZ} tests the dynamics.

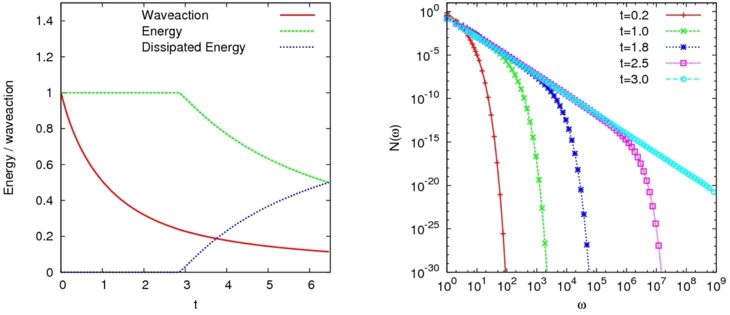
- Stationary scaling exponents are insufficient for validation.
- For some model interactions c_{KZ} can be calculated exactly.
- Product kernel:

$$L_1(\omega_1, \omega_2) = (\omega_1\, \omega_2)^{\frac{\lambda}{2}}.$$

- Sum kernel:

$$L_1(\omega_1, \omega_2) = \frac{1}{2}\left(\omega_1^{\lambda} + \omega_2\lambda\right).$$

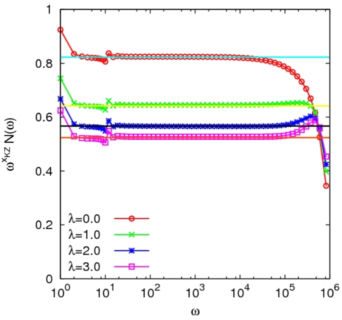
Example Results



Choices of spectral trunction

- It is necessary to truncate the calculation of collision integrals at $\omega = \Omega$: modes having $\omega > \Omega$ have $N_\omega = 0$.
- In sum over triads we only include $\omega_j \leq \omega_i < \omega_k \leq \Omega$.
- However we must *choose* what to do with triads having $\omega_j \leq \omega_i < \Omega < \omega_k$ (only relevant for $S_1[N_\omega]$).
- These terms are included in the sum with weighted by ν:
  - $\nu = 1$: open truncation (dissipative)
  - $\nu = 0$: closed truncation (conservative)
  - $0 < \nu < 1$: partially open truncation (dissipative)
- "Boundary conditions" on the energy flux are not local for integral collision operators.

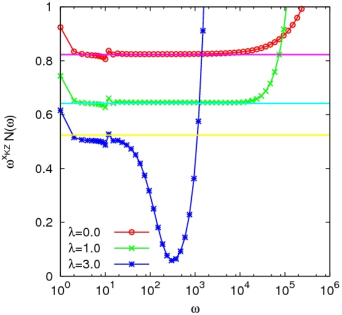
Open Truncation : $\nu = 1$ - Bottleneck Phenomenon



Compensated stationary spectra with open truncation.

- Product kernel: $L(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}$.
- Open truncation can produce a bottleneck as the solution approaches the dissipative cut-off (Falkovich 1994).
- Bottleneck does not occur for all $L(\omega_1, \omega_2)$.
- Energy flux at Ω is 1.

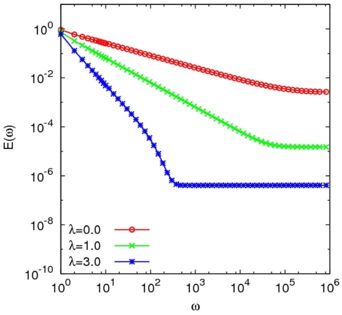
Closed Truncation: Thermalisation



Compensated quasi-stationary spectra with closed truncation.

- Product kernel: $L(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}$.
- Closed truncation produces thermalisation near the cut-off (CC and Nazarenko (2004), Cichowlas et al (2005)).
- Thermalisation occurs for all $L(\omega_1, \omega_2)$.
- Energy flux at Ω is 0.

Closed Truncation: Thermalisation



Bare quasi-stationary energy spectra with closed truncation.

- Product kernel: $L(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}$.
- Closed truncation produces thermalisation near the cut-off (CC and Nazarenko (2004), Cichowlas et al (2005)).
- Thermalisation occurs for all $L(\omega_1, \omega_2)$.
- Energy flux at Ω is 0.

Finite and Infinite Capacity Cascades

Stationary KZ spectrum:

$$N_\omega = c_{\rm KZ}\,\sqrt{J}\,\omega^{-\frac{\lambda+3}{2}}.$$

Total energy contained in the spectrum:

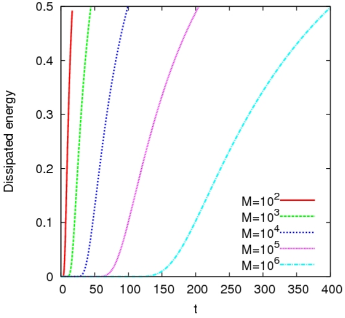
$$E = c_{\rm KZ}\,\sqrt{J}\,\int_1^\Omega d\omega\,\omega^{-\frac{\lambda+1}{2}}.$$

- E diverges as $\Omega \to \infty$ if $\lambda \leq 1$: *Infinite Capacity* .
- E finite as $\Omega \to \infty$ if $\lambda > 1$: *Finite Capacity* .

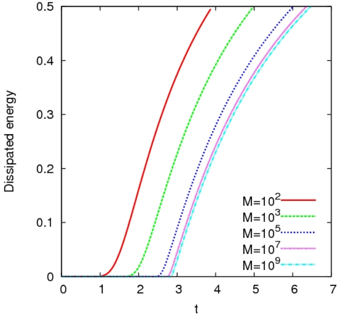
Transition occurs at $\lambda = 1$.

Dissipative Anomaly (decay problem, open truncation)

Finite capacity systems exhibit a dissipative anomaly in the usual sense:



$\lambda = 3/4$



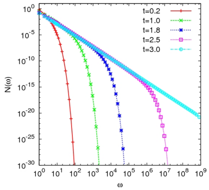
$\lambda = 3/2$

Dynamical Scaling

Self-similarity ansatz describing the establishment of the K–Z spectrum (Falkovich and Shafarenko, 1991) :

$$N(\omega, \tau) = \tau^a F(\eta)$$

propagating front with power law "wake".



$\eta = \frac{\omega}{\tau^b}$ $\tau = t$ Infinite capacity case
$\tau = t^* - t$ Finite capacity case

Dynamical scaling exponents, a and b.

$$x = -\frac{a}{b}$$

is the exponent of the wake.

Dynamical Scaling

Self-similarity requires : $a + (\lambda + 1)b = -1$.
Profile F determined by integro-differential equation:

$$\pm b\eta \frac{dF}{d\eta} - aF = S_1[F(\eta)] + S_2[F(\eta)] + S_3[F(\eta)]$$

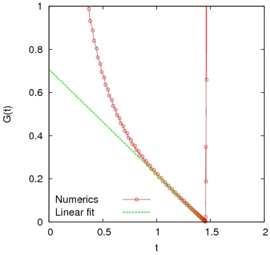
Infinite capacity case:

- Total energy $E \sim t \Rightarrow 2b + a = 1$.
- $a = \frac{\lambda+3}{\lambda-1}, \qquad b = -\frac{2}{\lambda-1}$
- $x = \frac{\lambda+3}{2}$ which is the K–Z exponent.

Finite capacity case:

- ?

Measuring dynamical scaling exponents



Measuring dynamical scaling via G_3.

- Finite capacity singularity $(t - t^*)^b$ makes direct fitting difficult.
- An easier measurement:

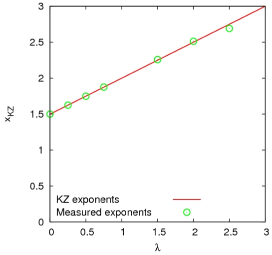
$$M_n(t) = \int \omega^n N_\omega$$

$$R_n(t) = \frac{M_{n+1}(t)}{M_n(t)}$$

$$G_n(t) = \frac{R_n(t)}{\dot{R}_n(t)}$$

Self-similarity ansatz $\Rightarrow G_n(t) \sim \frac{1}{b}(t - t^*)$. (fit a linear function).

Is there a dynamical scaling anomaly?



Dynamical scaling exponents for product kernel.

- Transient spectrum is often steeper than x_{KZ} for finite capacity cascades? (Galtier et al. (2000), Lee (2000), CC,Newell and Pomeau (2003), CC and Nazarenko (2004))
- This phenomenon is not well understood.
- If there is an anomaly in general, it is very small.

Cascades without back-scatter

The kinetic equation without backscatter:

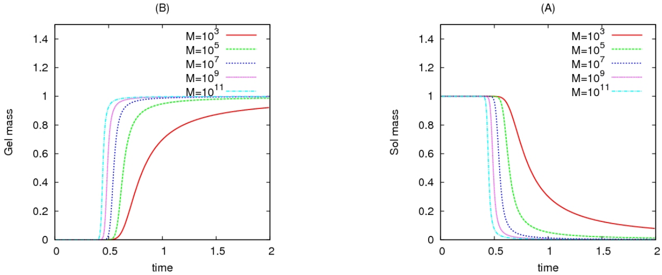
$$\frac{d\mathbf{N}}{dt} = S_1[\mathbf{N}].$$

What are the effects of removing backscatter?

- No thermalisation.
- Bottleneck phenomenon remains.
- A new type of singular solution emerges

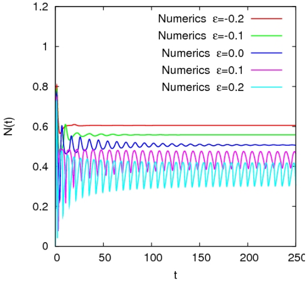
"Anomalous Dissipative Anomaly" (Dynamic Nonlocality)

$$\text{Decay problem :} \quad L_1(\omega_1, \omega_2) = \omega_1^\lambda + \omega_2^\lambda$$



For $\lambda > 1$, as $\Omega \to \infty$, $t^* \to 0$ removing *all* energy.

"Reset" Phenomenon and Nonlocal Oscillations



$$L(\omega_1, \omega_2) = \omega_1^{1+\epsilon} + \omega_2^{1+\epsilon}$$

- Dynamic nonlocality in the presence of a source leads to oscillatory behaviour.
- In such situations, there is no stationary state, no self-similarity.
- Unclear whether this phenomenon exists in the full 3WKE (can backscatter beat nonlocality?)

Conclusions

- New numerical method for solving the isotropic 3-wave kinetic equation
- Choice of spectral truncation allows one to produce bottleneck and / or thermalisation phenomena.
- Finite capacity systems exhibit a dissipative anomaly in the usual sense.
- Dynamical scaling exponents can be measured and do not show a strong dynamical scaling anomaly (at least for the product kernel).
- Removal of backscatter terms from the kinetic equation produces surprising new phenomena which suggest the scaling theory of the full kinetic equation may also contain hidden surprises.