Inertial waves dynamics

in rotating homogeneous turbulence

with F. Bellet, J. Scott, C. Cambon

F.S. Godeferd

Laboratoire de Mécanique des Fluides et d'Acoustique
École Centrale de Lyon, France

IHP "wave turbulence" workshop

Modelling inertial wave turbulence

Motivation : question about the full/partial two-dimensionalization of rotating turbulence at high rotation rate

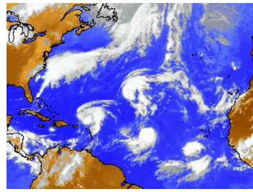
- Background - Linear and non linear regime
- Wave turbulence modelling : AQNM model – weakly nonlinear at asymptotically high rotation rate)
- Closure
- Dynamical equations for spectral tensors
- Numerical resolution
- Some results

(Ref. : JFM, 2006)

Background

$\longrightarrow$ Influence of solid body rotation on the structure of turbulence

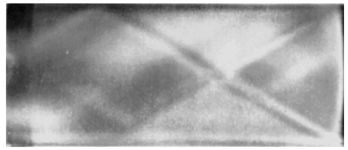
Areas :



- Geophysical flows
- Industrial flows : turbomachinery, hydraulic production of energy

Experimental landmarks :

⋆ Taylor (1921) $\longrightarrow$ 2D structuration in columns



⋆ McEwan (1970) $\longrightarrow$ characterization of inertial waves

Nonlinear waves interaction :

⋆ Benney & Saffman (1966) $\longrightarrow$ dynamique des amplitudes d'une distribution d'ondes

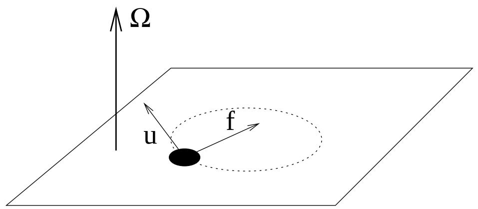
⋆ Zakharov & al. (1992) $\longrightarrow$ energy fluxes between waves

⋆ Caillol & Zeitlin (2000) kinetic equation for internal waves

Equations for rotating flows

Navier-Stokes equations in a rotating frame :[a]

$$(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u} + 2\Omega \boldsymbol{n} \times \boldsymbol{u} + \boldsymbol{\nabla} p - \nu \nabla^2 \boldsymbol{u} = 0$$



$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

modified pressure field

fluctuating velocity $\boldsymbol{u}$, pressure p

Non dimensional parameters :

$$Re = \frac{UL}{\nu} \equiv \frac{\text{nonlinear}}{\text{viscous}} \sim 10^6 \qquad Ro = \frac{U}{\Omega L} \equiv \frac{\text{nonlinear}}{\text{Coriolis}} \sim 0.1$$

$$Ek = \frac{Ro}{Re} \equiv \frac{\text{viscous}}{\text{Coriolis}}$$

[a] Equations for the complete velocity field are almost identical, with a modified pressure

The role of pressure **Inertial waves**

We take the linear inviscid limit for the rotating case at Ω.

– Without pressure, the linearized system admits sinusoidal solutions

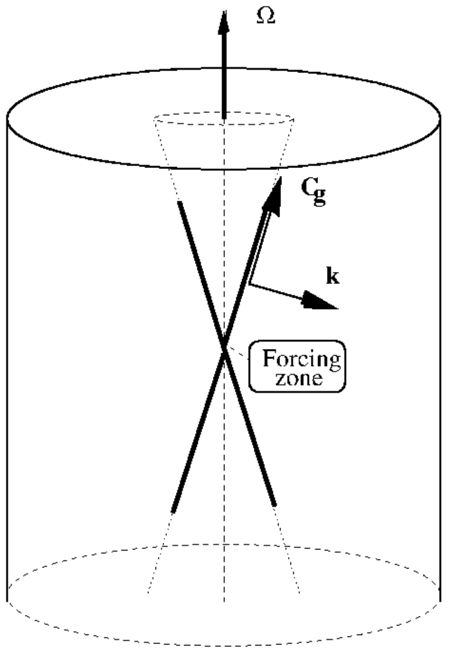
– These pressureless solutions are not valid in the general case, and p is needed to inforce the solenoidal property.

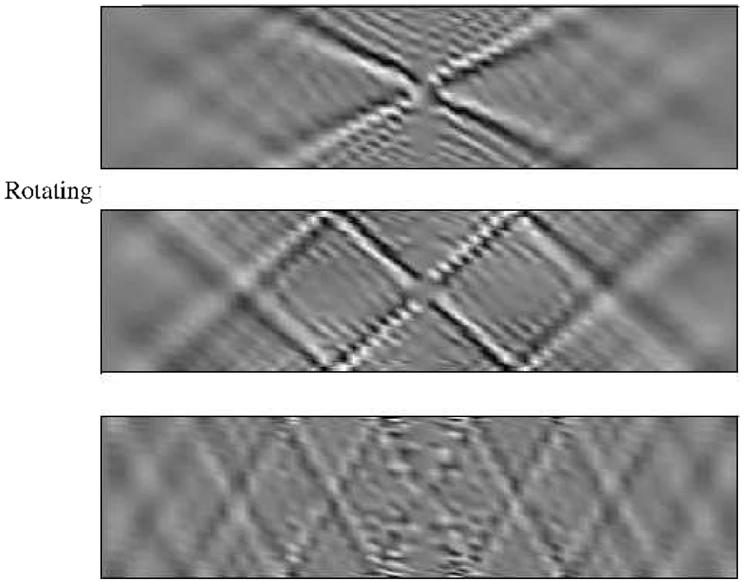
– Pressure is responsible for coupling horizontal and vertical velocity components, and for the anisotropic dispersion law of inertial waves $\Rightarrow$ without pressure, these are only *oscillating* solutions but not actual *propagating* waves.

Pressure equation obtained by eliminating $\boldsymbol{u}$ in the linearized system :

$$\partial_t^2 \left(\nabla^2 p\right) + (2\Omega)^2 \nabla_{\|}^2 p = 0$$

Illustration **Inertial waves**

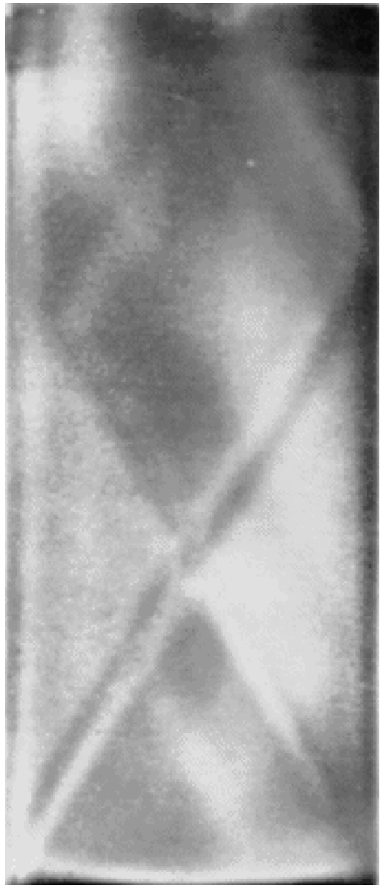




$$\frac{2\Omega}{n} = 1.10$$

$$\frac{2\Omega}{n} = 1.33$$

$$\frac{2\Omega}{n} = 2$$



Numerical simulation

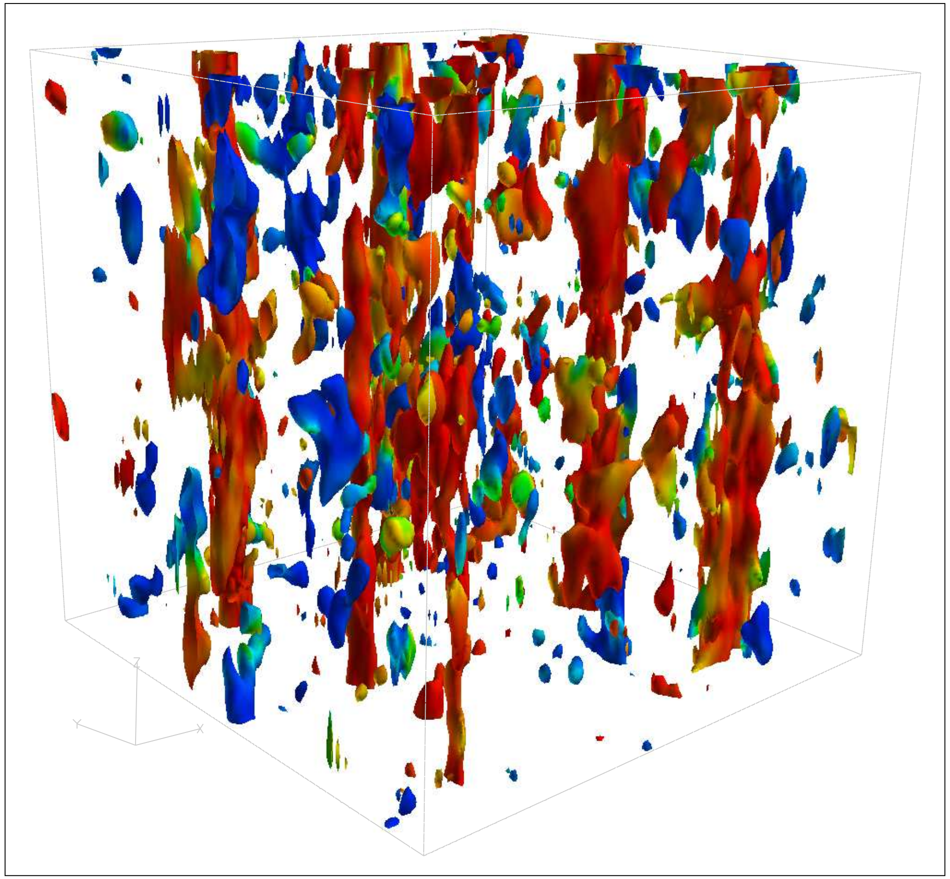
Godeferd & Lollini, 1999

Experiment

McEwan, 1970

Illustration

Rotating homogeneous turbulence



Iso-surfaces of vertical vorticity component (512^3 DNS by Liechtenstein, 2005)

Dominant linear effects

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \left[\mathsf{L}_\Omega + \mathsf{L}_\nu\right](\hat{\boldsymbol{u}}) = T(u,u) \quad \boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0$$

Linear regime : *Ro* $= u/(2\Omega L) \to 0$ $\qquad T(u,u) \to 0$

Linear solution : plane waves with constant amplitude $a \exp \mathsf{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \sigma t)$.

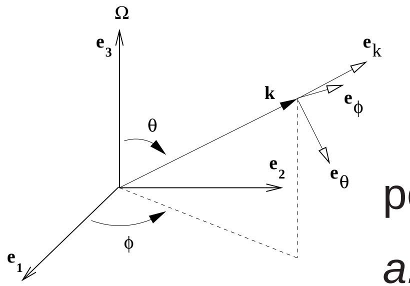
$$\hat{\boldsymbol{u}}(\boldsymbol{k},t) = \mathrm{e}^{-\nu \mathrm{k}^2 \mathrm{t}}\left[\mathrm{a}_{+1}\mathrm{e}^{\mathrm{i}\sigma\mathrm{t}}\mathrm{N} + \mathrm{a}_{-1}\mathrm{e}^{-\mathrm{i}\sigma\mathrm{t}}\mathrm{N}^*\right]$$

with

- $a_{+1}(\boldsymbol{k}), a_{-1}(\boldsymbol{k})$ — time independent amplitudes
- $\sigma(\boldsymbol{k}) = 2\Omega\cos\theta$ — anisotropic dispersion law $\neq k^\alpha$
- $\boldsymbol{N}(\boldsymbol{k}) = \boldsymbol{e}_1(\boldsymbol{k}) + \mathsf{i}\boldsymbol{e}_2(\boldsymbol{k})$ — eigenvector of $\mathsf{L} = \mathsf{L}_\Omega + \mathsf{L}_\nu$

Comparison with other dispersive waves

$$\begin{cases} \boldsymbol{e}_k(\boldsymbol{k}) = \boldsymbol{k}/k \\ \boldsymbol{e}_\theta(\boldsymbol{k}) = \boldsymbol{e}_\phi(\boldsymbol{k}) \times \boldsymbol{e}_k(\boldsymbol{k}) \\ \boldsymbol{e}_\phi(\boldsymbol{k}) = \boldsymbol{\Omega} \times \boldsymbol{k}/|\boldsymbol{\Omega} \times \boldsymbol{k}| \end{cases}$$



polar-spherical reference frame

a.k.a Craya-Herring frame

$$\partial_t \begin{pmatrix} \hat{u}^{(1)} \\ \hat{u}^{(2)} \\ \hat{T}^* \end{pmatrix} + \begin{pmatrix} \nu k^2 & -\sigma_r & 0 \\ \sigma_r & \nu k^2 & -\sigma_s \\ 0 & \sigma_s & \kappa k^2 \end{pmatrix} = 0$$

rotation+stratification : inertio-gravity waves

$$\partial_t \begin{pmatrix} \hat{u}^{(1)} \\ \hat{u}^{(2)} \\ \hat{b}^{(1)} \\ \hat{b}^{(2)} \end{pmatrix} + \begin{pmatrix} \nu k^2 & -\sigma_r & -\mathrm{i}\sigma_B & 0 \\ \sigma_r & \nu k^2 & 0 & \mathrm{i}\sigma_B \\ -\mathrm{i}\sigma_B & 0 & \eta k^2 & 0 \\ 0 & -\mathrm{i}\sigma_B & 0 & \eta k^2 \end{pmatrix} = 0$$

rotation+ext. magn. field : inertio-Alfvén waves

Wave turbulence

Weakly nonlinear dynamics :

a is a slow variable, such that the total amplitude is $e^{i\sigma t}a(\boldsymbol{k},t)$ i.e. *time dependent* amplitudes : $a_{+1}(\boldsymbol{k},t)$ et $a_{-1}(\boldsymbol{k},t)$

With $\dfrac{\partial \boldsymbol{u}}{\partial t} + \mathsf{L}(\boldsymbol{u}) = \mathsf{T}(\boldsymbol{u},\boldsymbol{u})$

$$\partial_t a_\epsilon(\boldsymbol{k},t) = \sum_{\epsilon',\epsilon''} \int_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \mathrm{e}^{\mathrm{i}\overbrace{[\epsilon\sigma(\boldsymbol{k}) + \epsilon'\sigma(\boldsymbol{p}) + \epsilon''\sigma(\boldsymbol{q})]}^{\mathrm{F}}\mathrm{t}} \mathrm{m}_{\epsilon\epsilon'\epsilon''}(\mathrm{k},\mathrm{p},\mathrm{q})$$

$$\times a_{\epsilon'}(\boldsymbol{p},t) a_{\epsilon''}(\boldsymbol{q},t)\, d^3\boldsymbol{q}$$

Triadic resonance : $F = 0$

Resonant surfaces

The above term $\mathrm{e}^{\mathrm{i}\left[\epsilon\sigma(\mathrm{k})+\epsilon'\sigma(\mathrm{p})+\epsilon''\sigma(\mathrm{q})\right]\mathrm{t}}$ allows for the following splitting of the interaction triads :

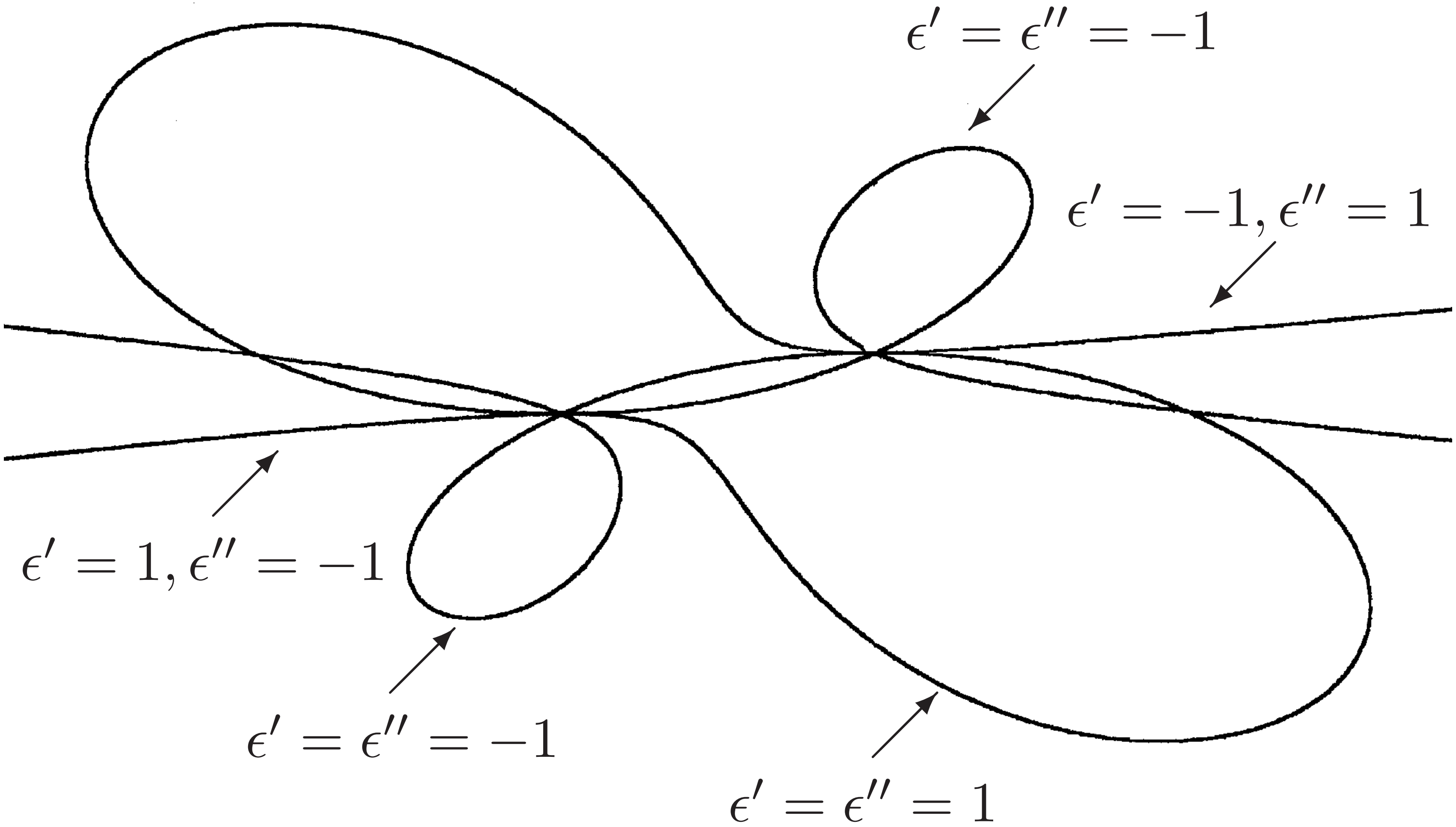
– Triads which satisfy the *resonance conditions* :

$$\begin{cases} \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0} \\ F_{\epsilon\epsilon'\epsilon''}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})=\epsilon\sigma(\boldsymbol{k})+\epsilon'\sigma(\boldsymbol{p})+\epsilon''\sigma(\boldsymbol{q})=0 \end{cases}$$

which defines complex *resonant surfaces* $S_{\epsilon'\epsilon''}$

– The remaining triads with $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}$ but $F_{\epsilon\epsilon'\epsilon''}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\neq 0$, in the 3D spectral space.

When rotation is fast, the phase of these remaining triads is a very rapidly oscillating term, so that their contribution is negligible. $\Rightarrow$ resonant triads are dominant in the dynamics of rapidly rotating turbulence.



Cut trough the resonant surface for $k = 1$ and $\theta_k = 1.3$

The path to a tractable model :

Solve numerically the equations, integrating only over the resonant surface, cheaper than full 3D spectral integration.

– Difficult issues when using random amplitude equations (the amplitudes oscillations are very fast, velocity fluctuations are not “smooth” quantities and yield inconsistencies in the Fourier representation)

– But the statistics, *i.e.* $< a_\epsilon a_{\epsilon'} >$ *are* smooth functions and one can use their equations as a starting point for modelling.

– A closure of the quasi-normal type has to be used (EDQNM⟶AQNM, Bellet 2003).

$\cdots$ + lots of analysis

Statistical approach

$\star$ Spectral tensor $\Phi_{ij} \sim \overline{uu}$ in the Craya-Herring frame : e energy density ; z polarization ; h helicity

$$\Phi_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e + z_r & z_i + ih \\ 0 & z_i - ih & e - z_r \end{pmatrix}$$

$\star$ Wave turbulence : $\Phi_{ij}(\boldsymbol{k}, t) = \sum_{s,s'} A_{ss'}(\boldsymbol{k}, t) N_i^{(s)}(\boldsymbol{k}) N_j^{(s')}(\boldsymbol{k}) e^{i(s+s')\omega(\boldsymbol{k})t}$

Three *equivalent* formalisms for the dynamics : (1) $\Phi_{ij} \sim \overline{uu}$; (2) $A_{ss'} \sim \overline{aa}$; (3) e, z et h

$\longrightarrow$ For instance, using $\overline{uu}$: infinite hierarchy of equations for the moments of order n :

$$\begin{aligned} \frac{\partial \overline{uu}}{\partial t} + \mathsf{L}^{(1)}(\overline{uu}) &= \mathsf{T}^{(1)}(\overline{uuu}) \\ \frac{\partial \overline{uuu}}{\partial t} + \mathsf{L}^{(2)}(\overline{uuu}) &= \mathsf{T}^{(2)}(\overline{uuuu}) \\ &\cdots \end{aligned}$$

Intermediate step : EDQNM closure (1/2)

Two-point closure truncated at moments of order 4. Orszag, 1970. Adapted to include external distorsion through Green's tensor G.

Hypotheses :

- Quasi-Normal (QN) : $\overline{uuuu} = \sum \overline{uu}\ \ \overline{uu} \rightarrow$ closure, but $e < 0$
- Eddy-Damping (ED) : correct memory of $\overline{uuu}$ by damping $\zeta \rightarrow k^{-5/3}$ in isotropic turb.

EDQNM closure (2/2)

- Markovian (M) : distinguish rapid and slow evolutions (realizability $e < 0$)
  - ⋆ EDQNM1 : ζ rapid ; G and (e, z, h) slow $\longrightarrow$ no more nonlinear dynamics due to Ω
  - ⋆ EDQNM2 : ζ and G rapid ; (e, z, h) slow $\longrightarrow$ not valid at large $|z|$
  - ⋆ EDQNM3 : ζ, G and oscillating part of z rapid ; (e, Z, h) slow

$\Rightarrow$ coupled closed equations for e, Z et h. For instance

$$\frac{\partial e}{\partial t} + 2\nu k^2 e = T^e$$

with

$$T^e = \sum_{s',s''} \int_{\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}''=0} \Re\left(\frac{b_{s's''}}{\zeta^{\text{total}} + iF_{1s's''}}\right) e'' \left[e' - e\right] \mathrm{d}^3\boldsymbol{k}'$$

$+$ other terms involving Z and h

AQNM model — Hypotheses (1/3)

To overcome some of the EDQNM3 limitations :

- Moderate spatial discretization
- No specific treatment of resonant triads (surfaces)

$\longrightarrow$ asymptotique model for *Ro* $\ll 1$

Additional hypotheses :

$\star$ Nonlinear dynamics at $t \gg \Omega^{-1} \Longleftrightarrow \Omega t \gg 1$

$\star$ Damping $\zeta \ll \Omega$

} consequences of *Ro* $\ll 1$

$\star$ *Re* $\longrightarrow \infty$: but viscosity can be re-introduced later

$\longrightarrow$ temporal evolution of $A_{ss'}$ and byproducts e, Z and h

AQNM model — Mathematical steps (2/3)

Three steps for the multiscale asymptotic development :

1. Use EDQNM3 for the nonlinear transfer, with markovianization of slow amplitudes $A_{ss'}$:

$$\frac{\partial A}{\partial t} = \mathsf{T}_{\text{EDQNM3}}(A, A)$$

$\longrightarrow$ simplification of nonlinear term

2. Elimate rapidly oscillating terms in $\boldsymbol{k}'$ (Ωt large)

$$\longrightarrow \int \frac{\zeta}{\zeta^2 + F^2_{1s's''}} (\cdots)\, \mathrm{d}^3\boldsymbol{k}' \text{ and } \int \frac{F_{1s's''}}{\zeta^2 + F^2_{1s's''}} (\cdots)\, \mathrm{d}^3\boldsymbol{k}'$$

3. Limit $\zeta \longrightarrow 0$:

- $\dfrac{\zeta}{\zeta^2 + F^2_{1s's''}} \longrightarrow \pi\delta(F_{1s's''})$, whence :

  $$\int \frac{\zeta}{\zeta^2 + F^2_{1s's''}} (\cdots)\, \mathrm{d}^3\boldsymbol{k}' \longrightarrow \pi \int_{F_{1s's''}=0} \frac{1}{|\nabla F_{1s's''}|} (\cdots)\, \mathrm{d}^2 S$$

  $\Longrightarrow$ integrals over the sole resonant surfaces

- $$\int \frac{F_{1s's''}}{\zeta^2 + F^2_{1s's''}} (\cdots)\, \mathrm{d}^3\boldsymbol{k}' \longrightarrow ⨍ \frac{1}{F_{1s's''}} (\cdots)\, \mathrm{d}^3\boldsymbol{k}'$$

  $\Longrightarrow$ principal value integrals

AQNM model — Final equations (3/3)

For kinetic energy spectrum e, polarization spectrum Z, helicity spectrum h :

- $$T^e = \sum_{s',s''} \int_{\substack{\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}''=0 \\ F_{1s's''}=0}} \frac{g_{s's''}}{\alpha_{s's''}} \left[e'' \left(e' - e\right) + s'h' \left(s''h'' - h\right)\right] \mathrm{d}^2 S$$

$$\alpha_{s's''} = \frac{1}{\pi} \left| s''\boldsymbol{C}_g(\boldsymbol{k''}) - s'\boldsymbol{C}_g(\boldsymbol{k}')\right| \text{ with } \boldsymbol{C}_g \text{ group velocity}$$

- $$T^Z = -Z \sum_{s',s''} \left[\int_{\substack{\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}''=0 \\ F_{1s's''}=0}} \frac{g_{s's''}}{\alpha_{s's''}} e' \, \mathrm{d}^2 S + i \fint_{\mathbb{R}^3} \frac{g_{s's''}}{F_{s's''}} e' \, \mathrm{d}^3 \boldsymbol{k}' \right]$$

- $$T^h = \sum_{s',s''} \int_{\substack{\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}''=0 \\ F_{1s's''}=0}} \frac{g_{s's''}}{\alpha_{s's''}} \left[s'h' \left(e'' - e\right) + e' \left(s''h'' - h\right)\right] \mathrm{d}^2 S$$

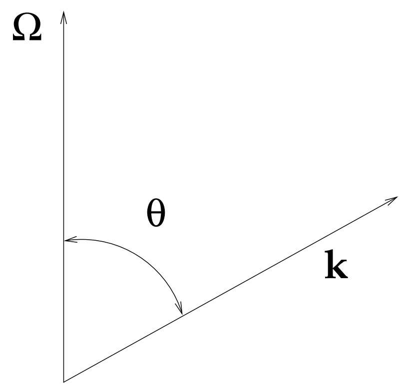
Remarks :

⋆ T^e is conservative and the model is realizable $\Rightarrow \forall t,\, e(t) \geq 0$

⋆ For initially isotropic turbulence without helicity/polarization $\Rightarrow \forall t,\, Z(t) = h(t) = 0$

⋆ ζ does not appear anymore

AQNM numerically solved equation



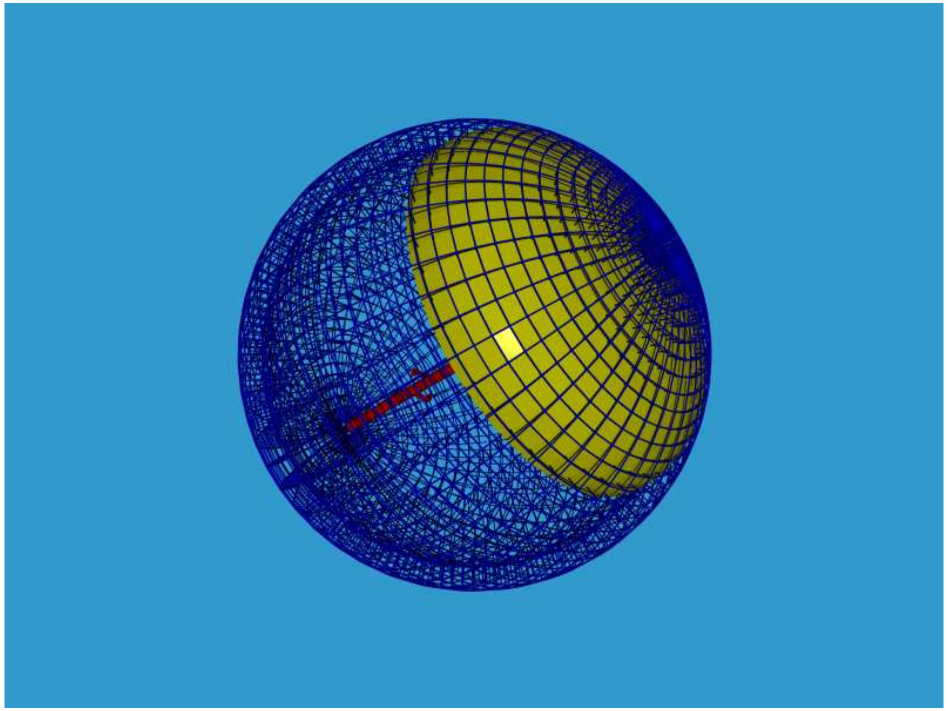
Energy equation $e(k,\theta)$:

$$\frac{\partial e}{\partial t} + 2\nu k^2 e = \sum_{\epsilon'\epsilon''} \int_{S_{\epsilon'\epsilon''}} \frac{g_{\epsilon'\epsilon''}(k,p,q)}{\alpha_{\epsilon'\epsilon''}(\boldsymbol{p},\boldsymbol{q})} e(\boldsymbol{p},t)\left[e(\boldsymbol{q},t) - e(\boldsymbol{k},t)\right] d^2\boldsymbol{p}$$

$$\alpha_{\epsilon'\epsilon''}(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{\pi}\left|\epsilon' \boldsymbol{c}_g(\boldsymbol{p}) - \epsilon'' \boldsymbol{c}_g(\boldsymbol{q})\right|$$

No explicit inclusion of exactly 2D modes

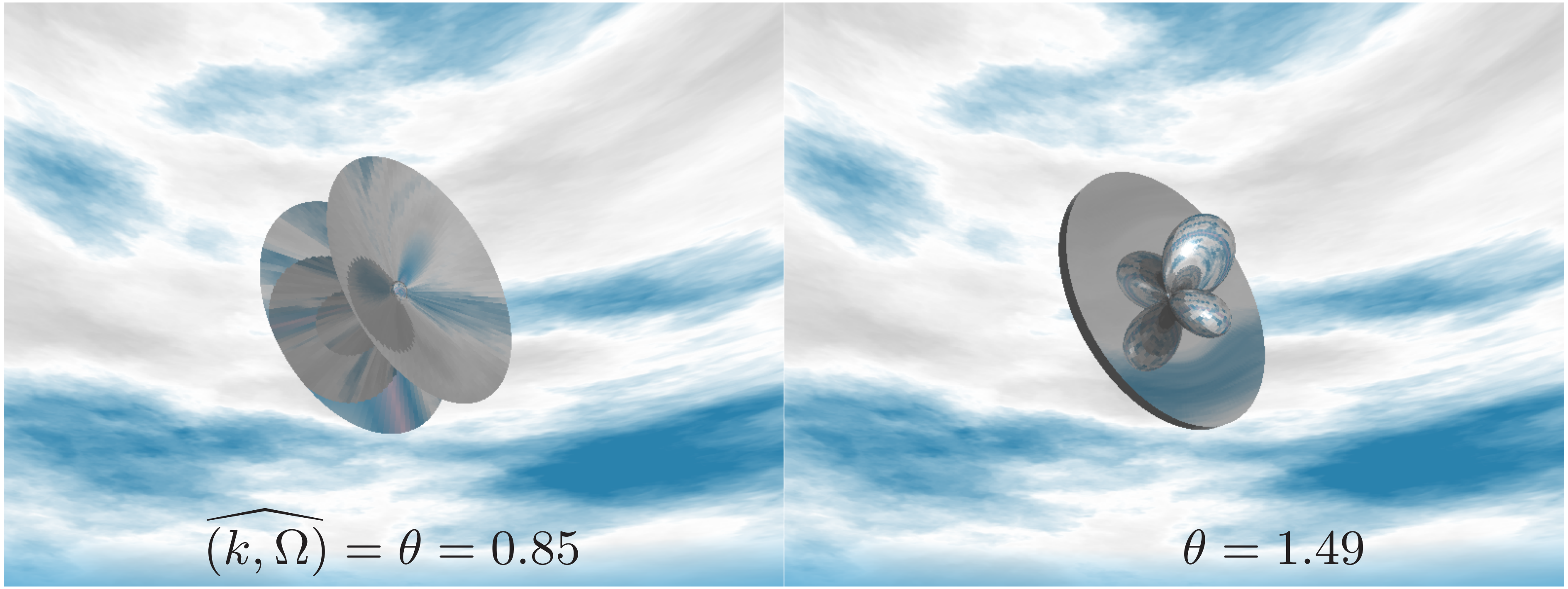
Numerical resolution



– Spherical discretisation of spectral space : Typical resolution $400 \times 400 \times 400$=16 million points (parallel computation)

– Compute intersection of resonant surface with each grid cell $\Rightarrow$ elementary area and integration geometrical coefficients

– 3D interpolation of spectrum for $\boldsymbol{q}$


$\widehat{(k,\Omega)} = \theta = 0.85$
$\theta = 1.49$

AQNM run

Initial isotropic conditions with narrow band spectrum

Need to stabilize the numerical scheme by re-introducing some viscosity : virtual Reynolds number $\mathfrak{R} = 5$ (truncation,bottleneck)

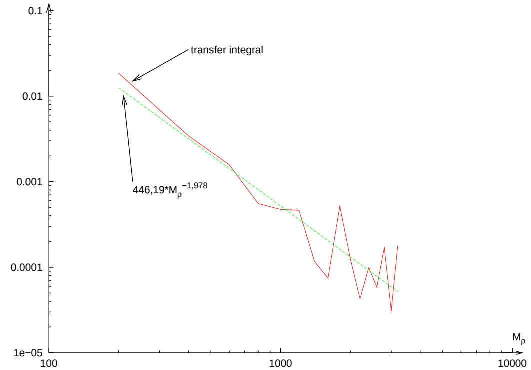
Unsteady run from $t_0 = 0$ to $t_f = 1,05$ (scaled by *Ro*$^{-2}\Omega^{-1}$)

A *unique* AQNM run is needed :

⋆ No dependency on *Ro* or *Re*

⋆ Universality of the non dimensional results

M_ρ convergence :



We study

⋆ Loss of isotropy $\longrightarrow$ two-dimensionalization ?

⋆ Inertial range scaling $\longrightarrow k^{-3}$ or k^{-2} power law ?

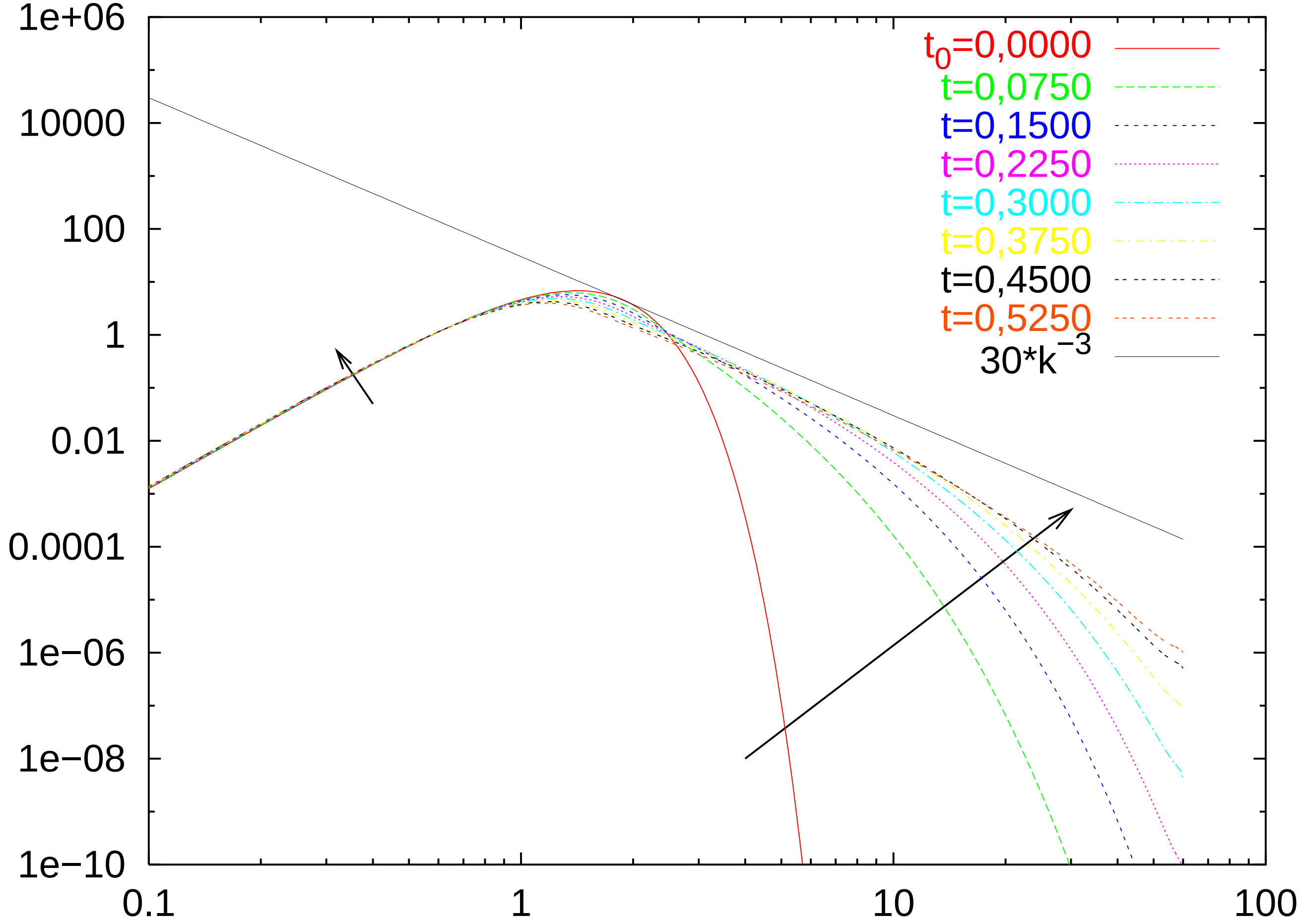
⋆ Rate of energy decay ?

AQNM : shell averaged energy spectrum

$E(k)$

Inertial range

set up



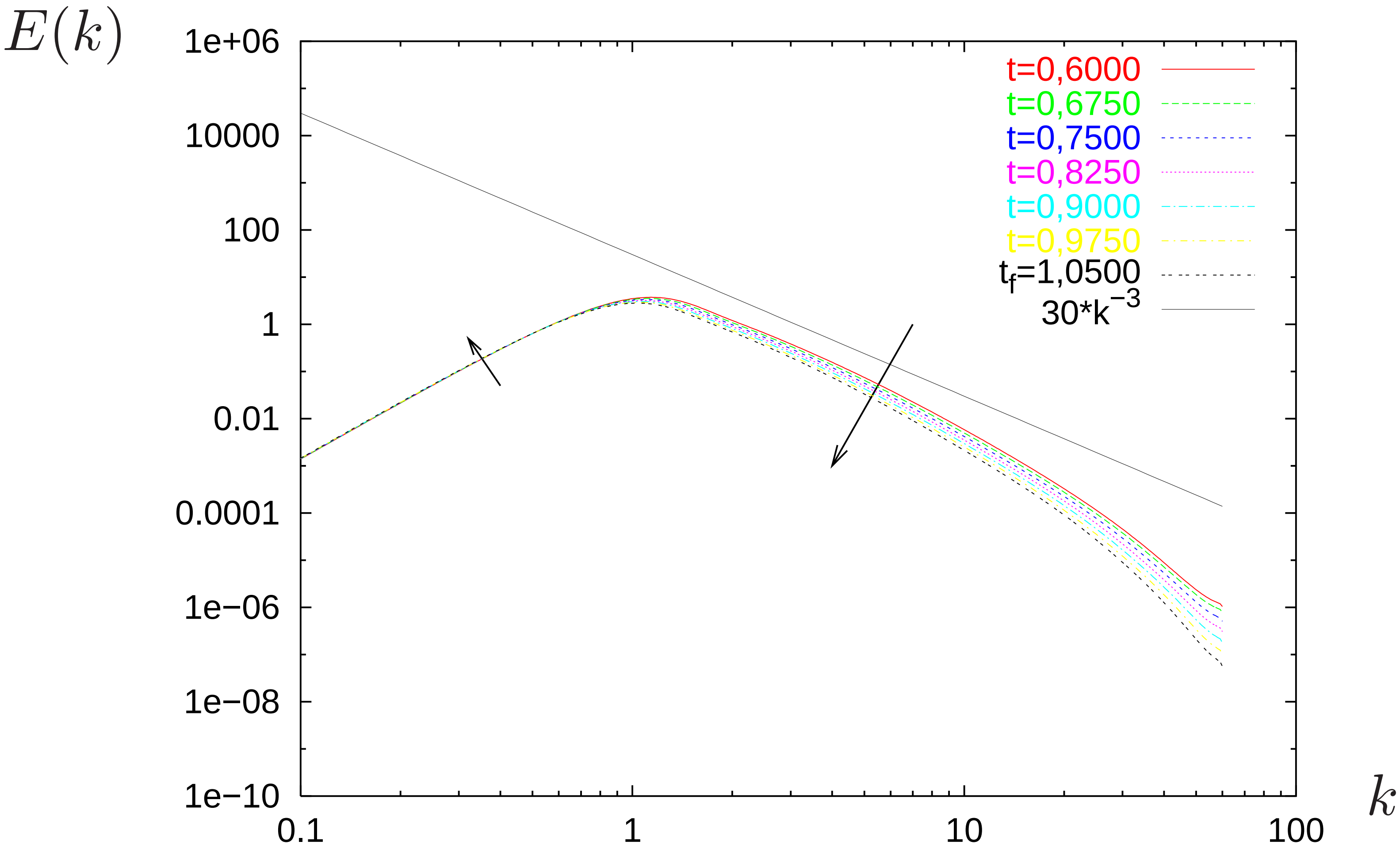
k

Kinetic Energy $E(k)$ such that $\int_{\mathbb{R}} E(k)\mathrm{d}k = \mathcal{E}$ (total energy)

Shell averaged energy spectrum

Viscous

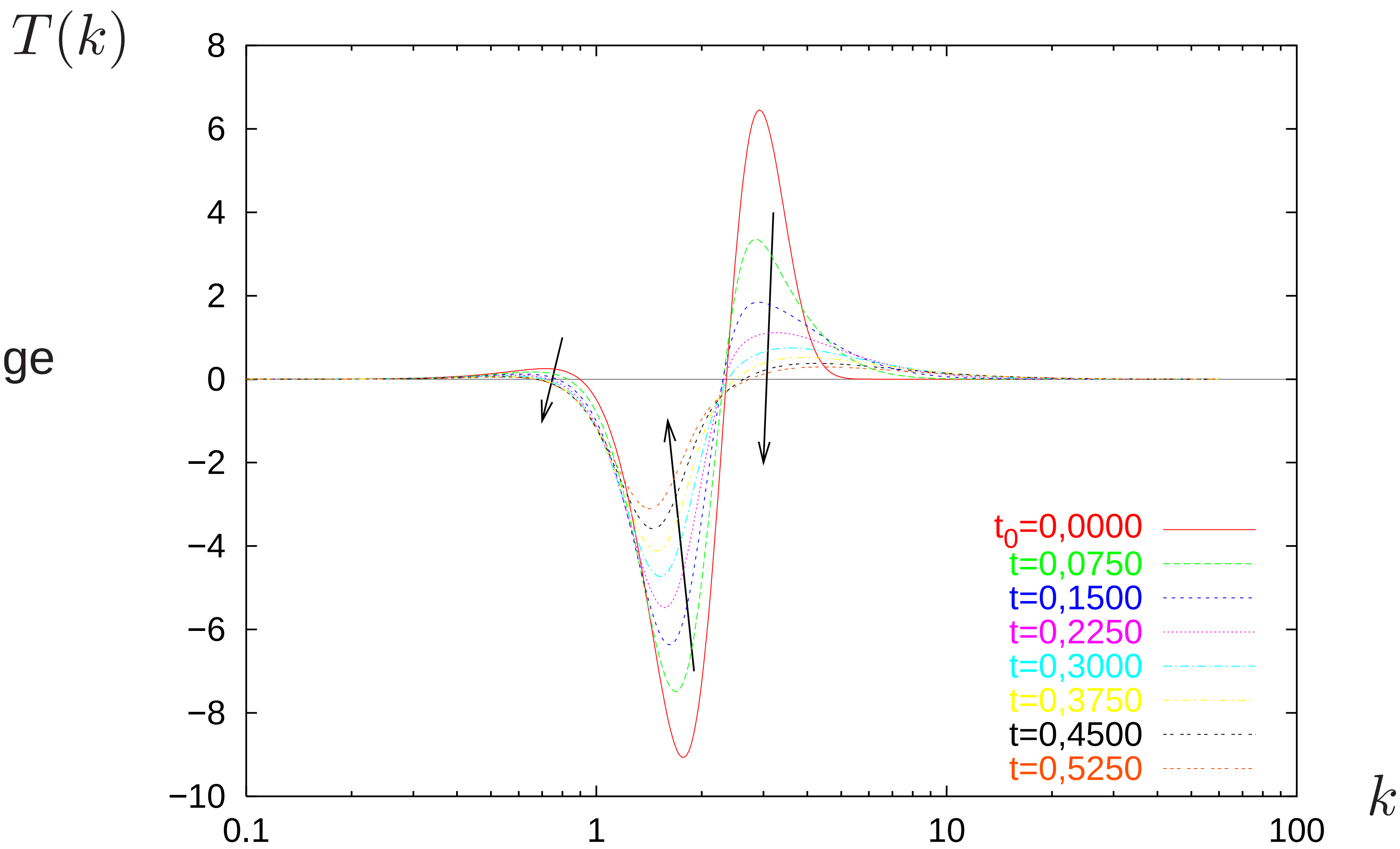
decay



Shell averaged energy transfer spectrum

Inertial range

set up

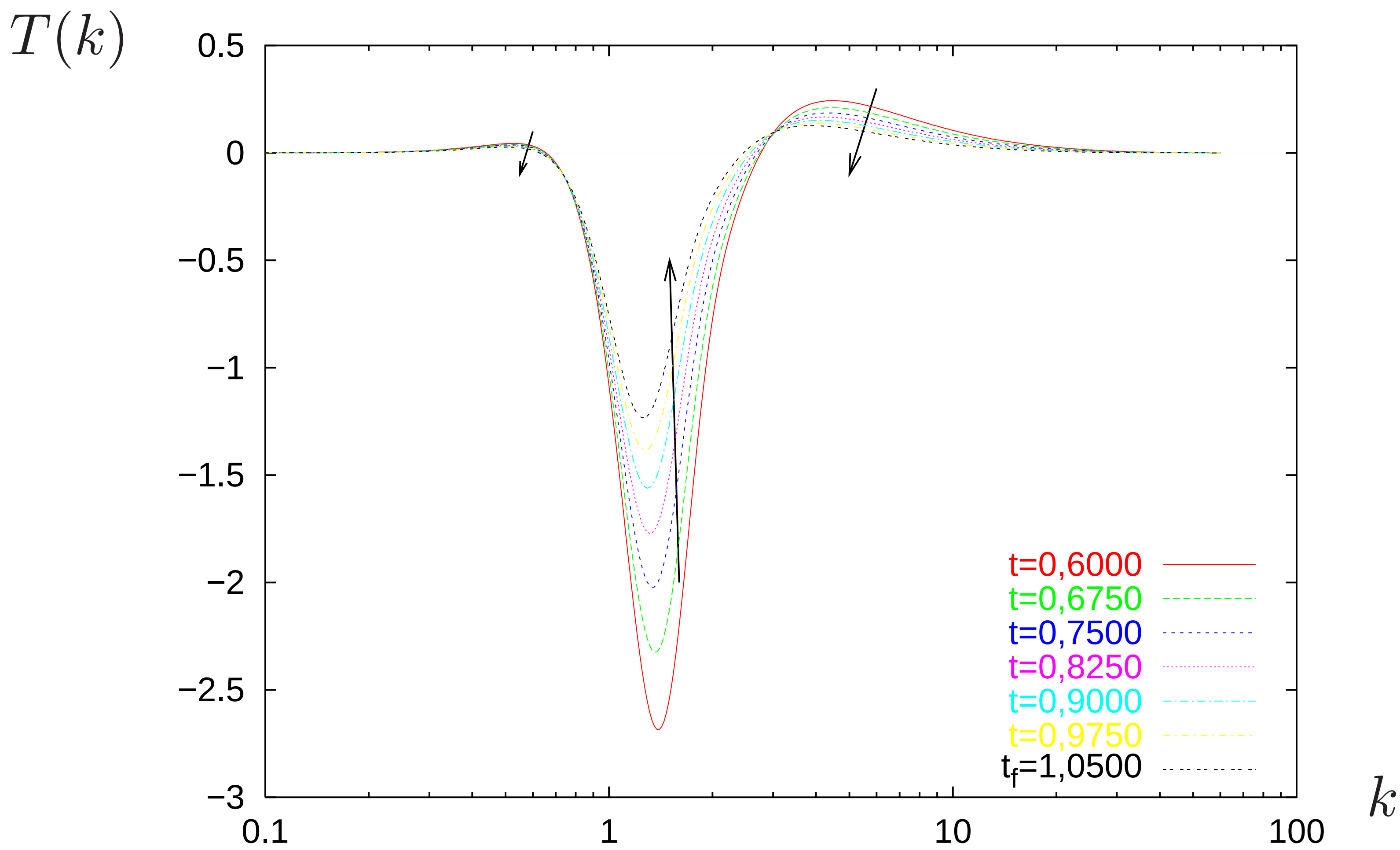


$T(k)$ such that $\int_{\mathbb{R}} T(k)\mathrm{d}k = 0$

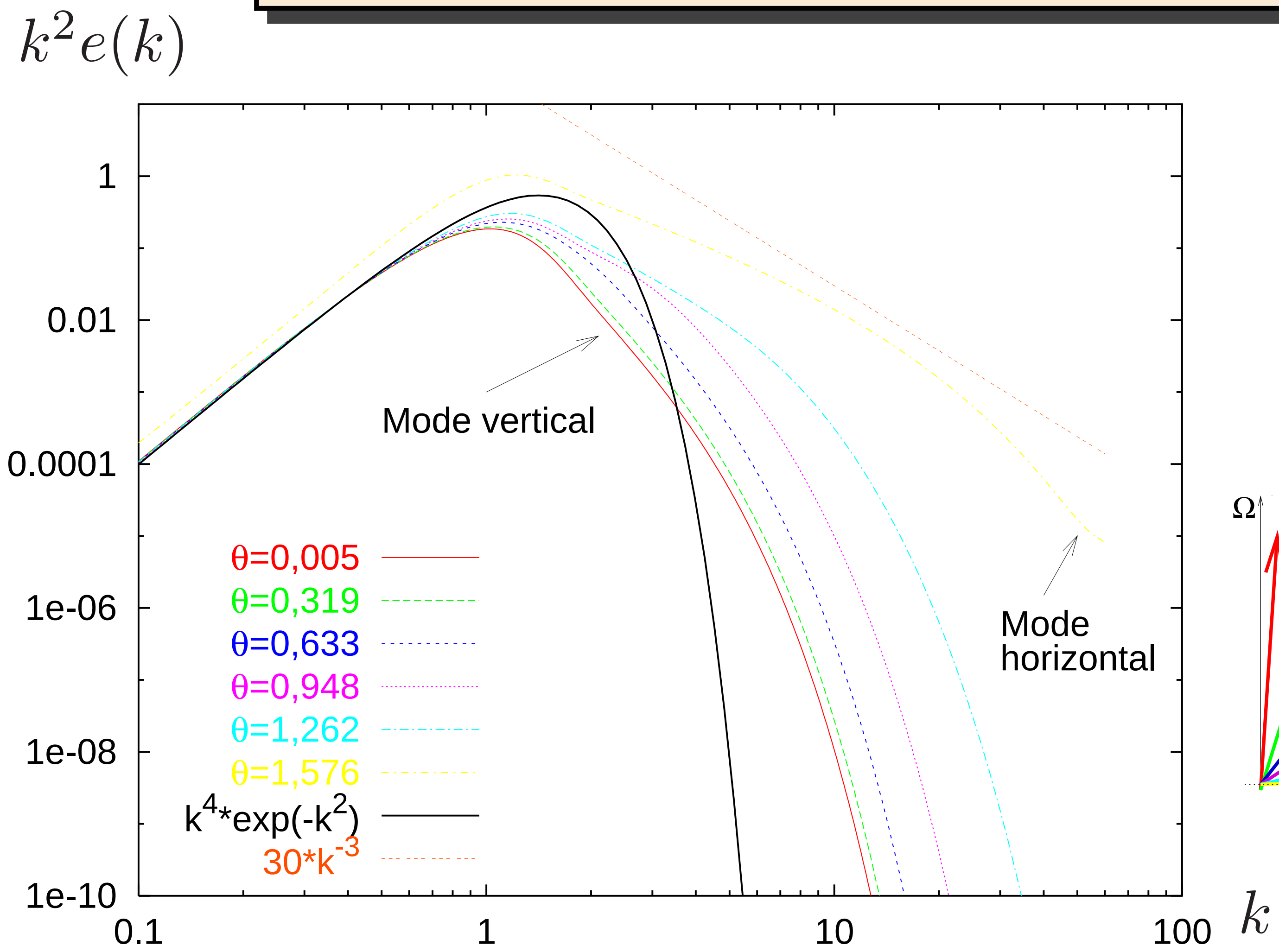
Shell averaged energy transfer spectrum

Viscous

decay

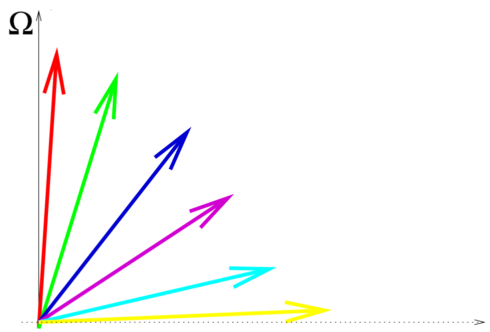


Azimuthal averaged energy density spectrum



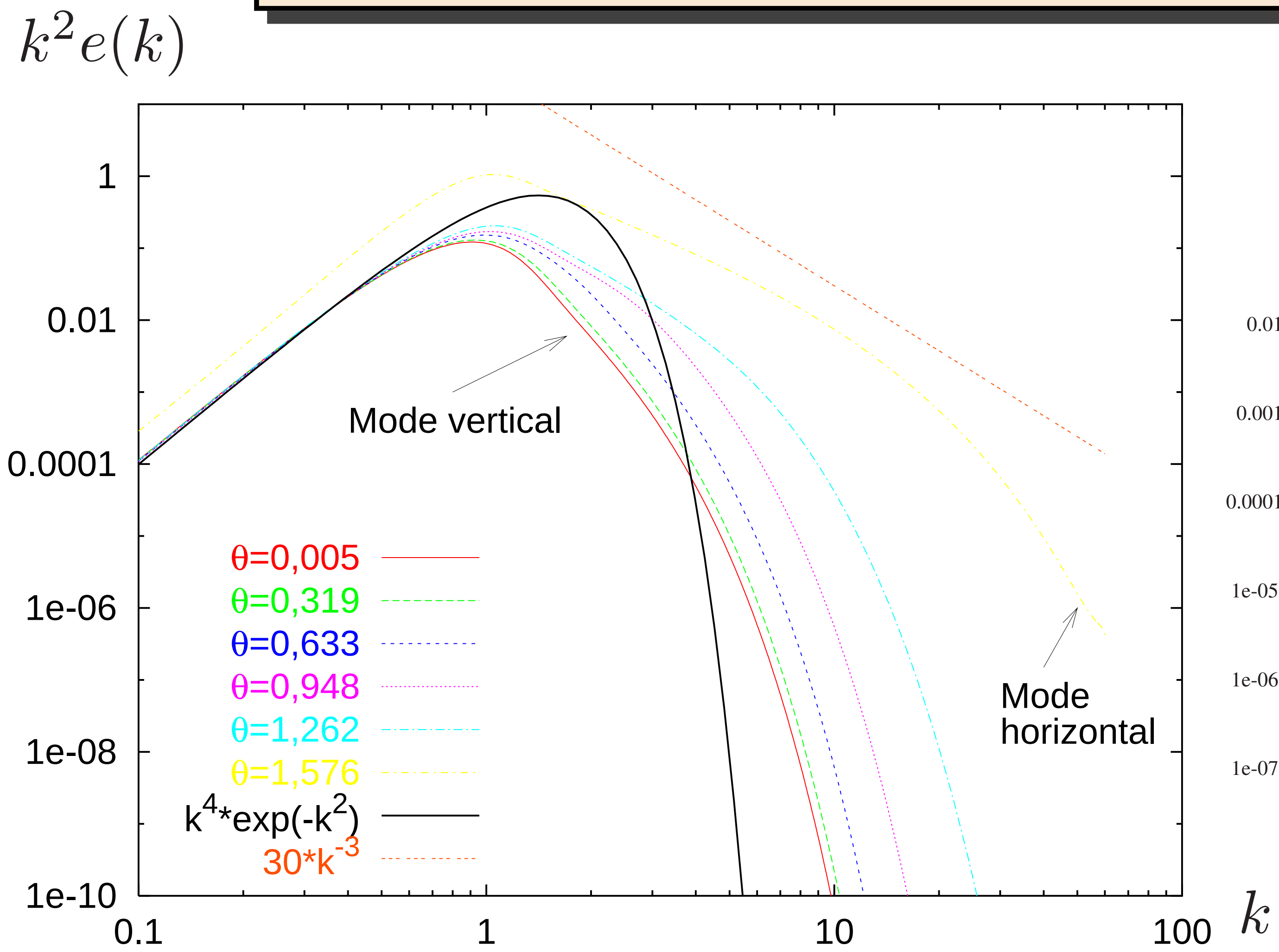
$t = 0,525$

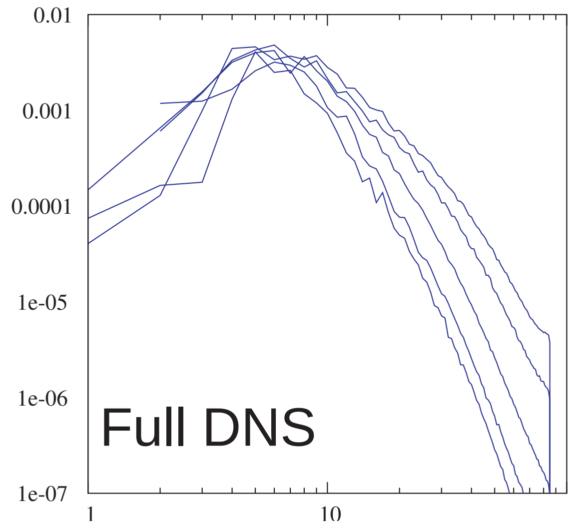
(inertial range established)



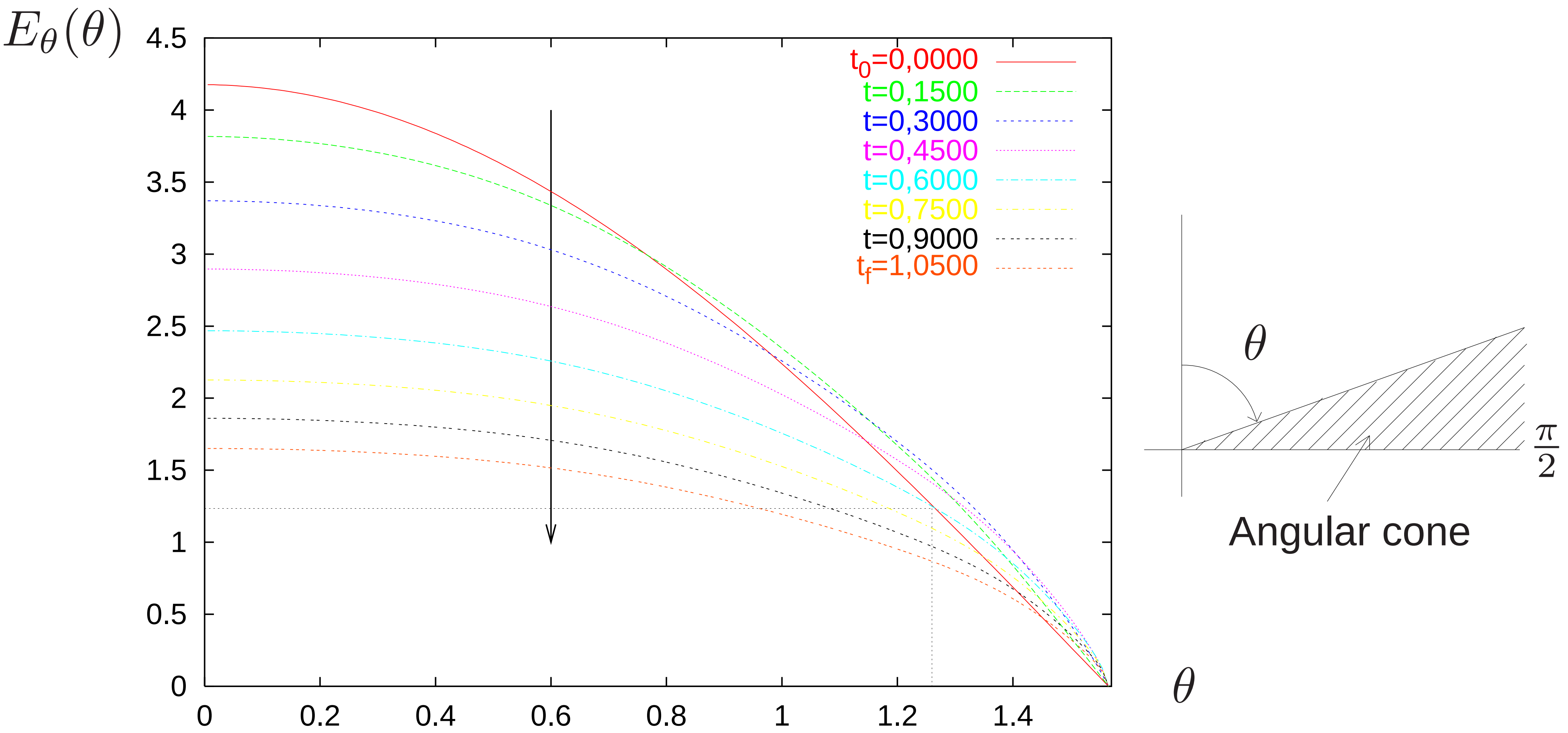
Azimuthal averaged energy density spectrum

$t_f = 1,05$





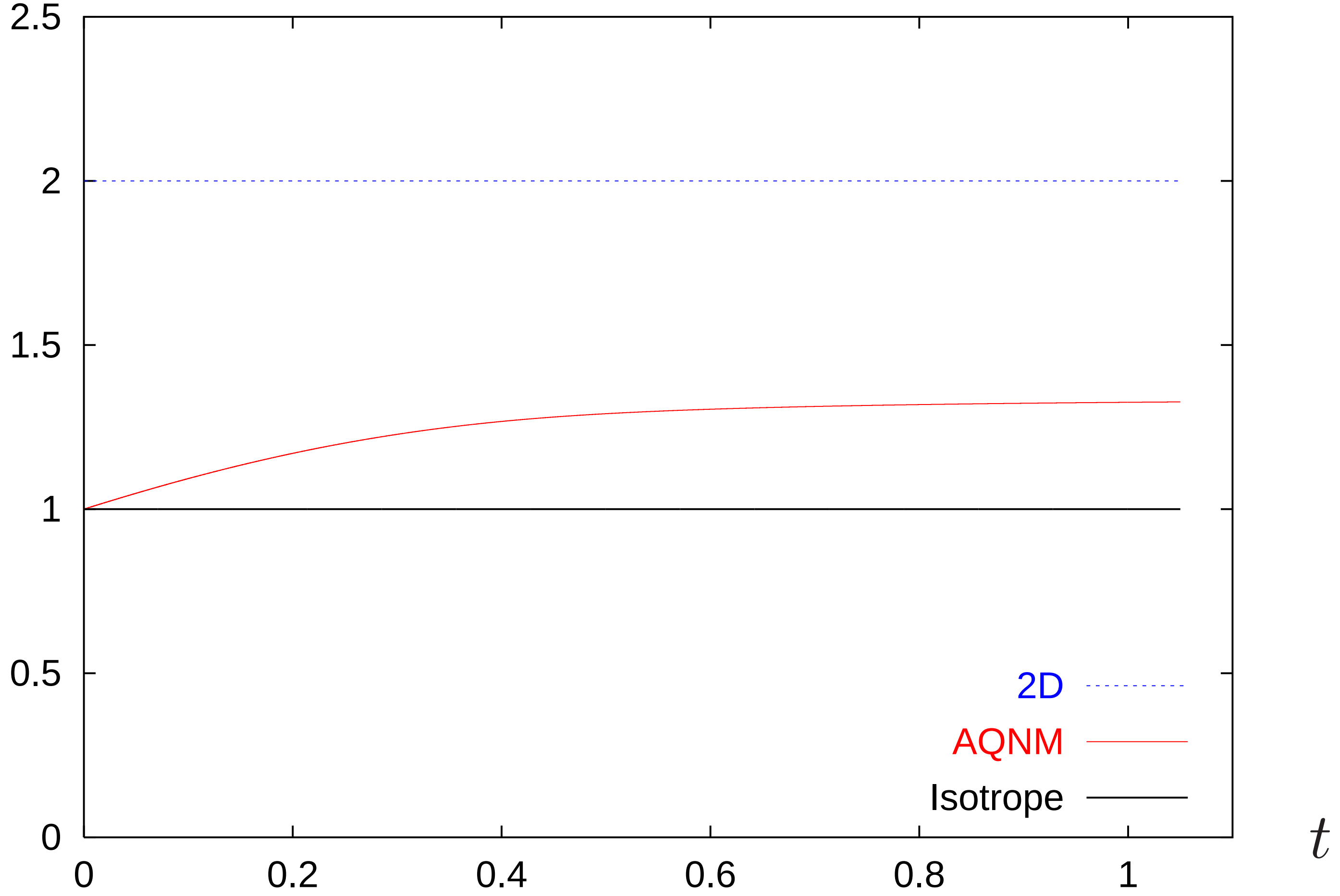
Angular energy



Energy E_θ between θ and $\pi/2$ at different times

Reynolds stress tensor anisotropy

$$\left(\frac{{u'_3}^2}{{u'_1}^2}\right)(t)$$



Summary

The AQNM model for wave turbulence :

⋆ Asymptotic in time model for small Rossby number (large rotation rate)

⋆ To our knowledge, first numerical resolution (tough job...)

⋆ *Explicit* expression of $e(\boldsymbol{k}, t)$ from the selected resonant interactions

Results :

1. Anisotropy created by rotation with stronger vertical coherence
2. Nonlinear transition towards two-dimensional state but not quite
3. Transfer of energy from rapid to slow modes
4. Reduced decay rate of turbulence $(\mathcal{E} \sim t^{-0,8})$
5. k^{-3} inertial range results from integration over *all* modes orientations
6. Model still lacks the matching between AQNM (rapid modes) and the exact two-dimensional manifold (2D turbulence part)