WAVE TURBULENCE IN THE DIRECT CASCADE OF THE 3D GROSS-PITAEVSKII EQUATION

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OUTLINE

- Short description of the GPE
- The effect of a condensate
- Numerical experiments
- Results

THE GROSS-PITAEVSKII EQUATION*
(Defocusing Nonlinear Schroedinger Equation)

$$i\frac{\partial\psi}{\partial t} + \nabla^2\psi - |\psi|^2\psi = F + D$$

- Optical waves
- Bose Einstein condensates
- Superfluid model

Linear dispersion relation

$$\omega(k) = k^2$$

* Gross, E.P. (May 1961). "Structure of a quantized vortex in boson systems". Il Nuovo Cimento 20 (3): 454-457.
Pitaevskii, L. P. (1961). "Vortex Lines in an Imperfect Bose Gas". Soviet Physics JETP 13 (2): 451-454

WAVE KINETIC EQUATION AND SOLUTIONS

$$\frac{\partial n_1}{\partial t} = \int n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) d\mathbf{k}_{2,3,4}$$

Collision invariants:

$$E = \int \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k} \qquad \mathbf{M} = \int \mathbf{k} n_{\mathbf{k}} d\mathbf{k} \qquad \mathrm{N} = \int N_{\mathbf{k}} d\mathbf{k}$$

Total energy | Total momentum | Total wave action

STATIONARY NONEQUILIBRIUM SOLUTIONS

$n_{\mathbf{k}} \sim k^{-3}$ direct energy cascade

$n_{\mathbf{k}} \sim k^{-7/3}$ inverse wave action cascade

$n_k^{(1D)} \sim k^{-1}$

direct cascade

THE BOGOLIUBOV DISPERSION RELATION

$$\psi = C_0 + \phi \qquad \phi << C_0$$

C_0 **condensate amplitude**

ϕ **fluctuations on the condensate**

Three wave system:

$$\omega_B(k) = |C_0|^2 + k\sqrt{k^2 + 2|C_0|^2}$$

SOME REFERENCES ON NUMERICAL SIMULATIONS OF GPE

2D:

Dyachenko, Newell, Pushkarev, Zakharov, Physca D, 1992

Dyachenko, Falkovich Physical Review E, 1996

Nazarenko and M.O., Physica D, 2006

3D:

Nore, Abid, Brachet, Physics of Fluids, 1997

Kobayashi, Tsubota, Journal of Physical Society of Japan, 2005

Yoshida, Armitsu, Journal of Low Temperature Physics, 2006

COMPUTATIONAL DETAILS

Pseudo spectral method

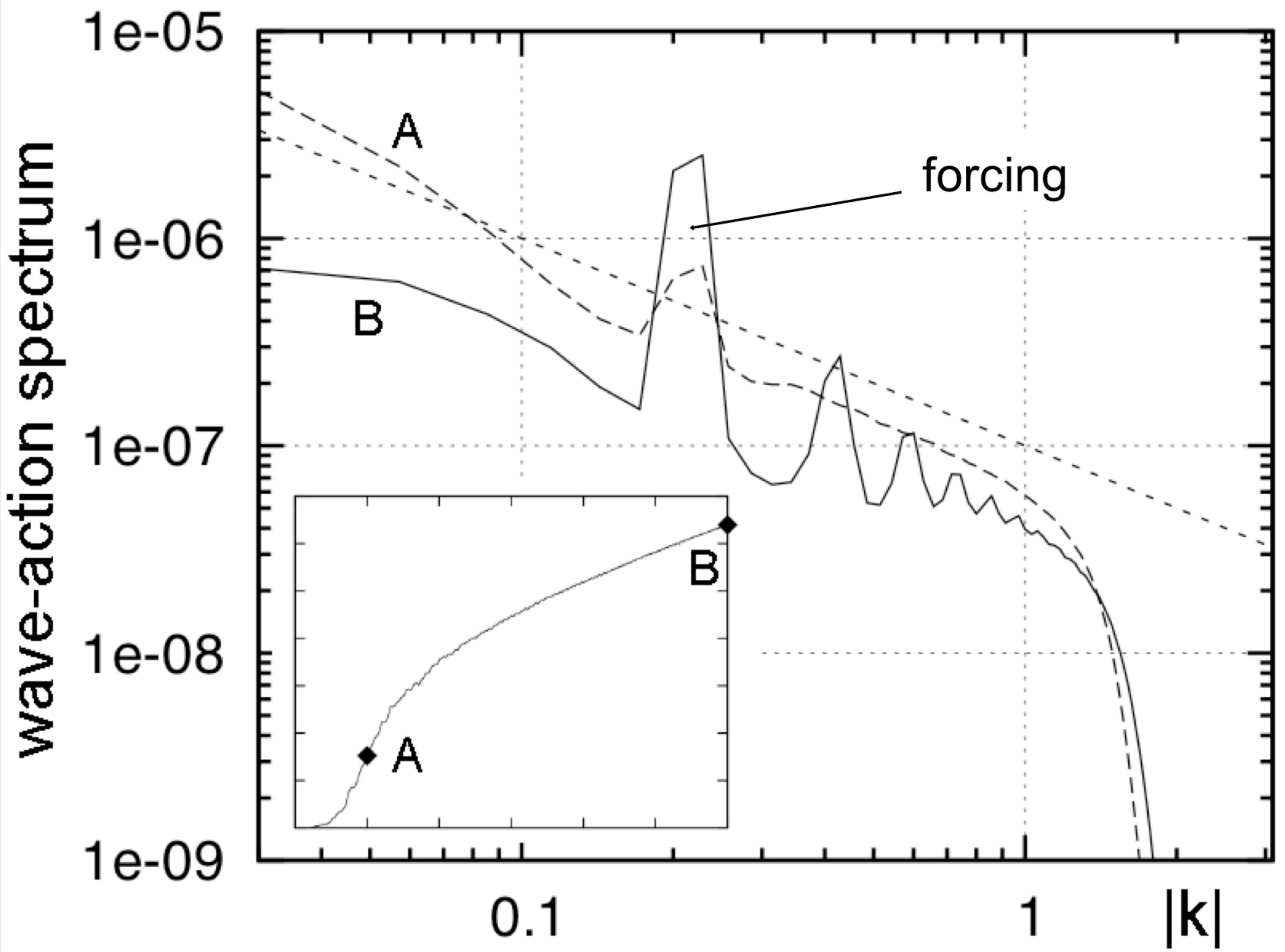
Periodic box 256^3 with $\Delta x = 1$

HIGH FREQUENCY DISSIPATION

$$D = i\nu_h (\nabla^2)^n \psi(\mathbf{x},t) \qquad \text{with } n = 8, \;\; \nu_h = 2\times 10^{-6}$$

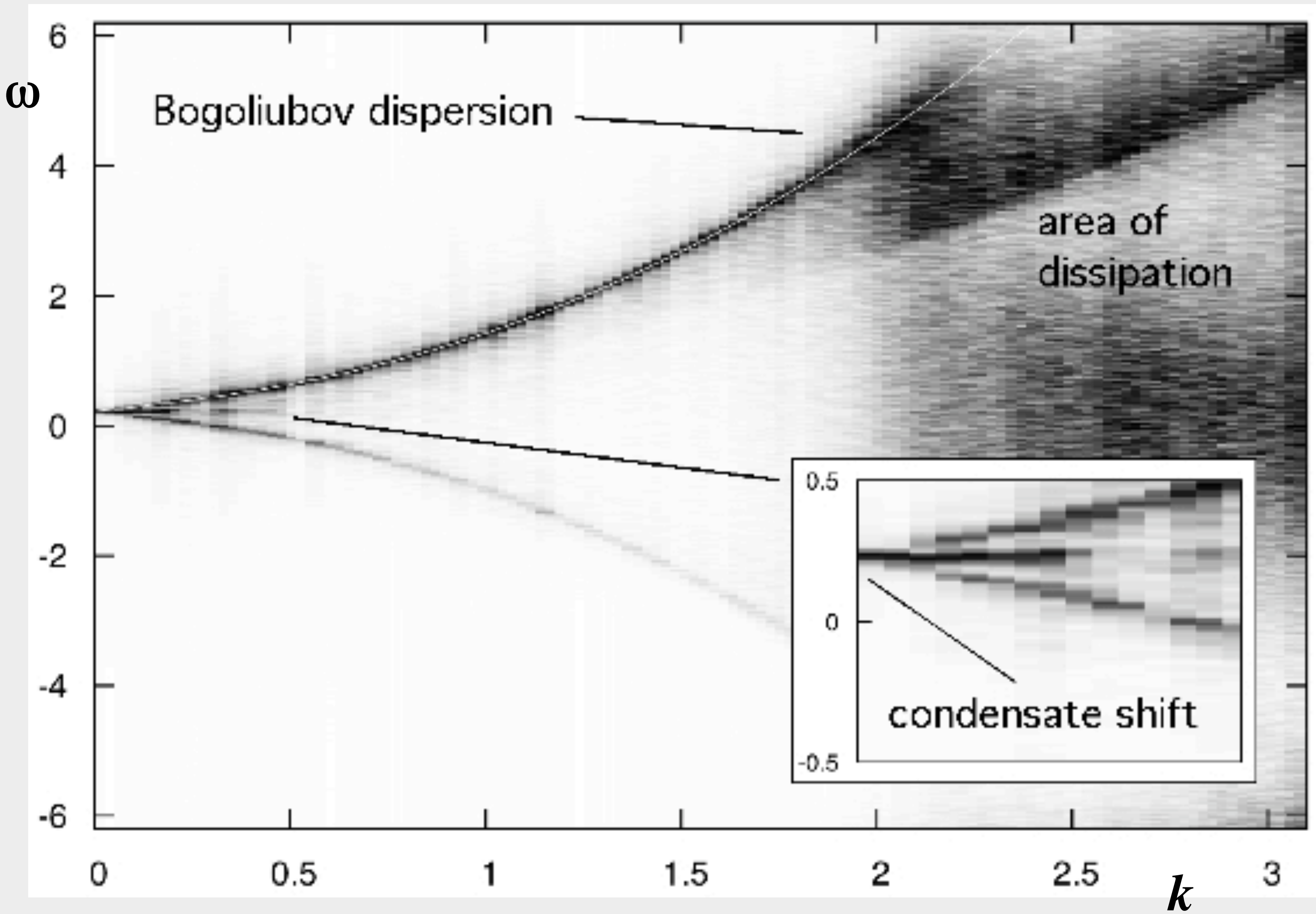
FORCING

$$\begin{cases} F = -i f_0 Exp[i\varphi(k,t)] & 8\Delta k < k < 11\,\Delta k \\ F = 0 & k \le 8\,\Delta k \text{ and } k \ge 11\,\Delta k \end{cases}$$



1e-05
1e-06
1e-07
1e-08
1e-09
wave-action spectrum
0.1
1
|k|
A
B
forcing
B
A

BOGOLIUBOV DISPERSION RELATION



DISSIPATION AT LOW WAVE NUMBERS

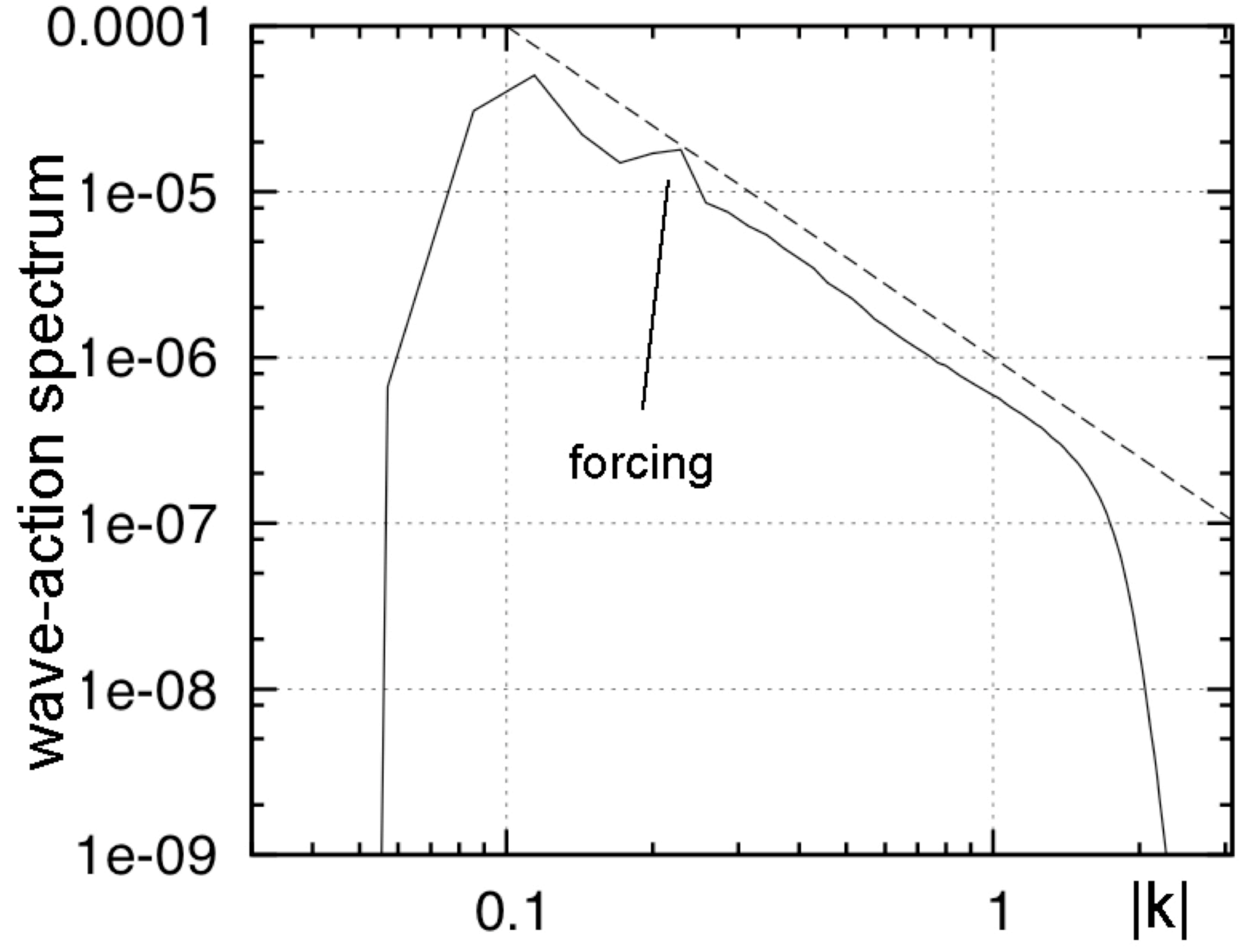
hypoviscosity

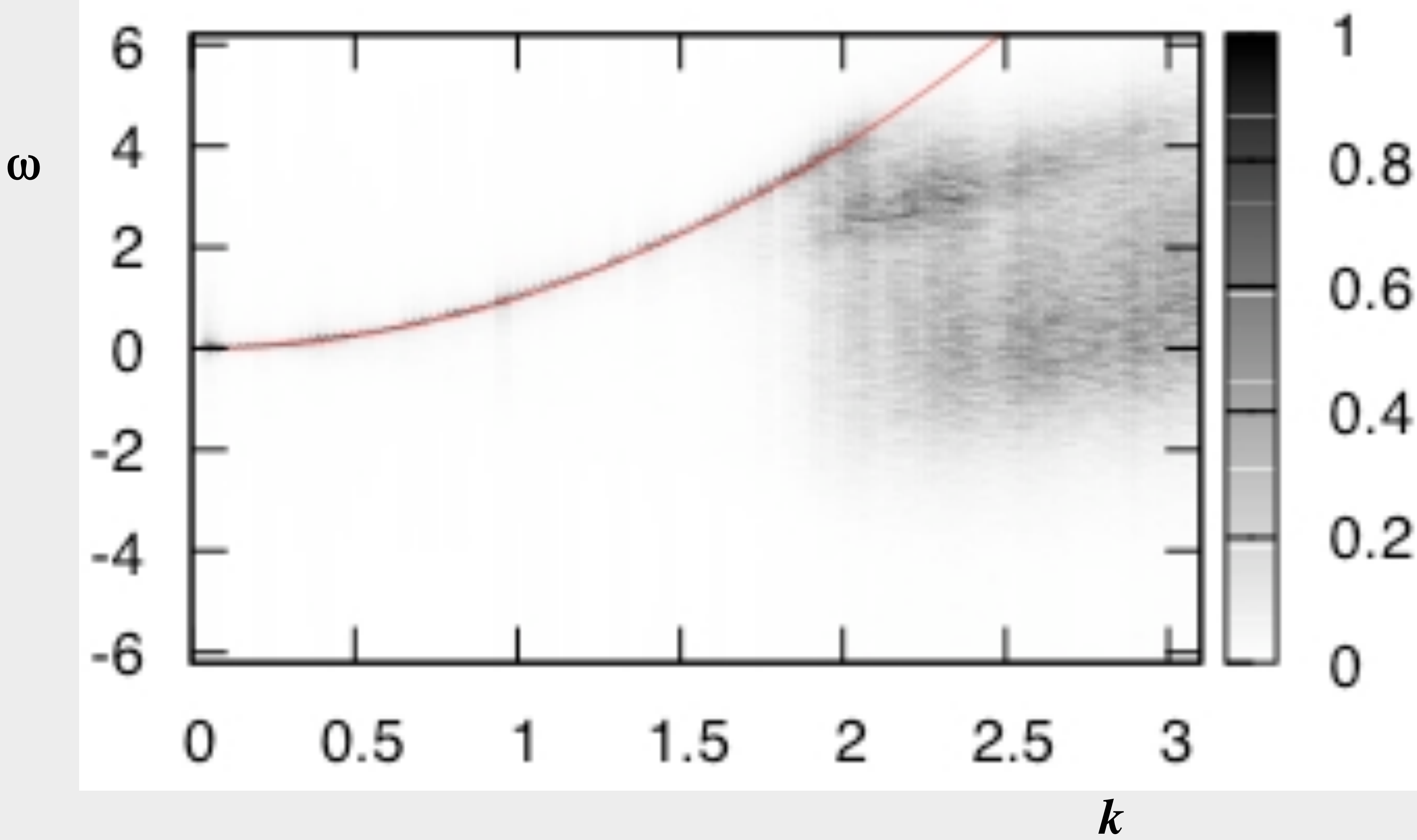
$$D = i\nu_l(\nabla^{-2})^m \psi(\mathbf{x},t) \qquad \text{with } m = 8, \;\; \nu_h = 1\times 10^{-18}$$

friction

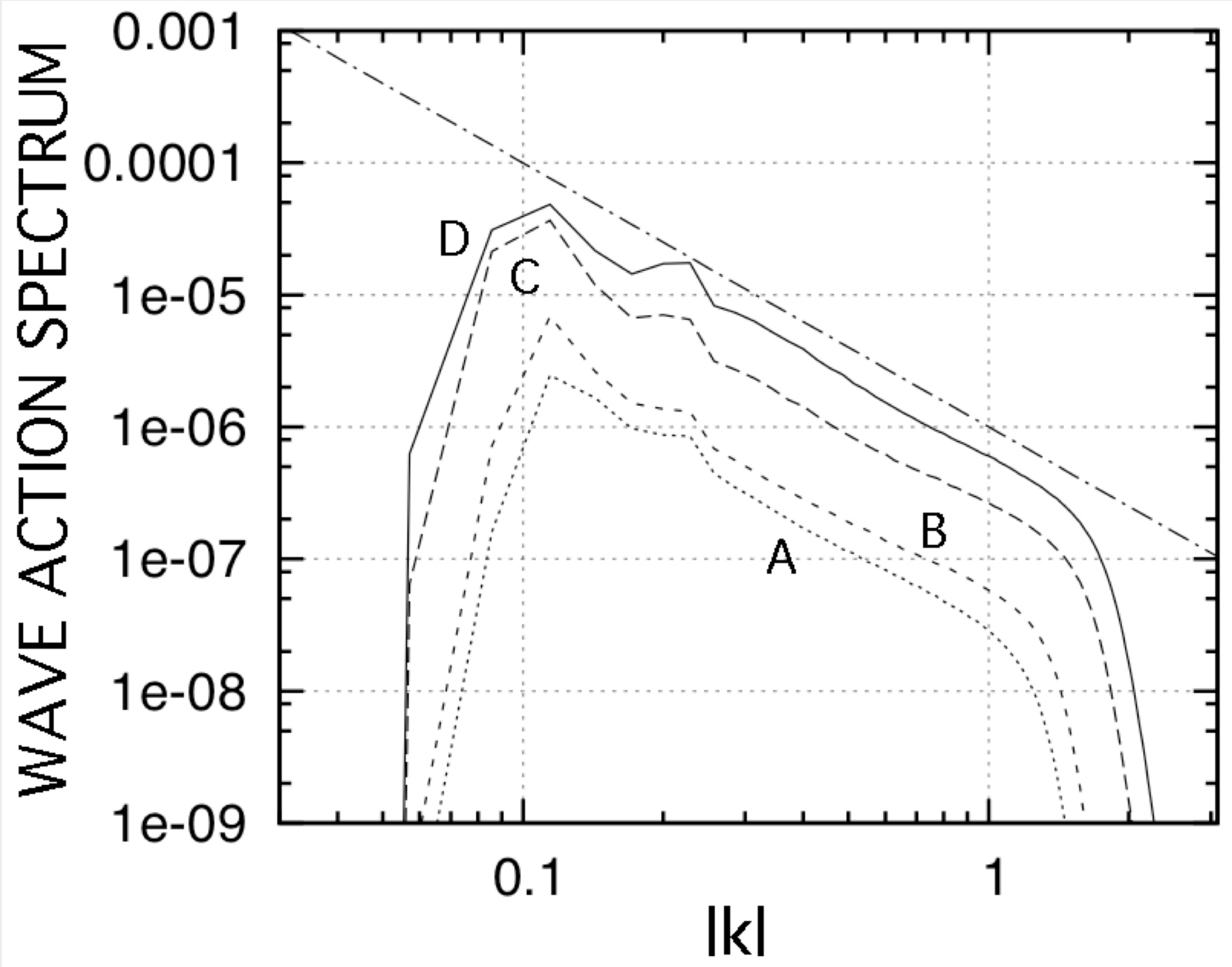
$$D = i\mu\theta(k_c - k)\psi(\mathbf{x},t) \qquad \text{with } k_c = 9\Delta k, \;\; \mu = 1\times 10^{-4}$$

With hypoviscosity

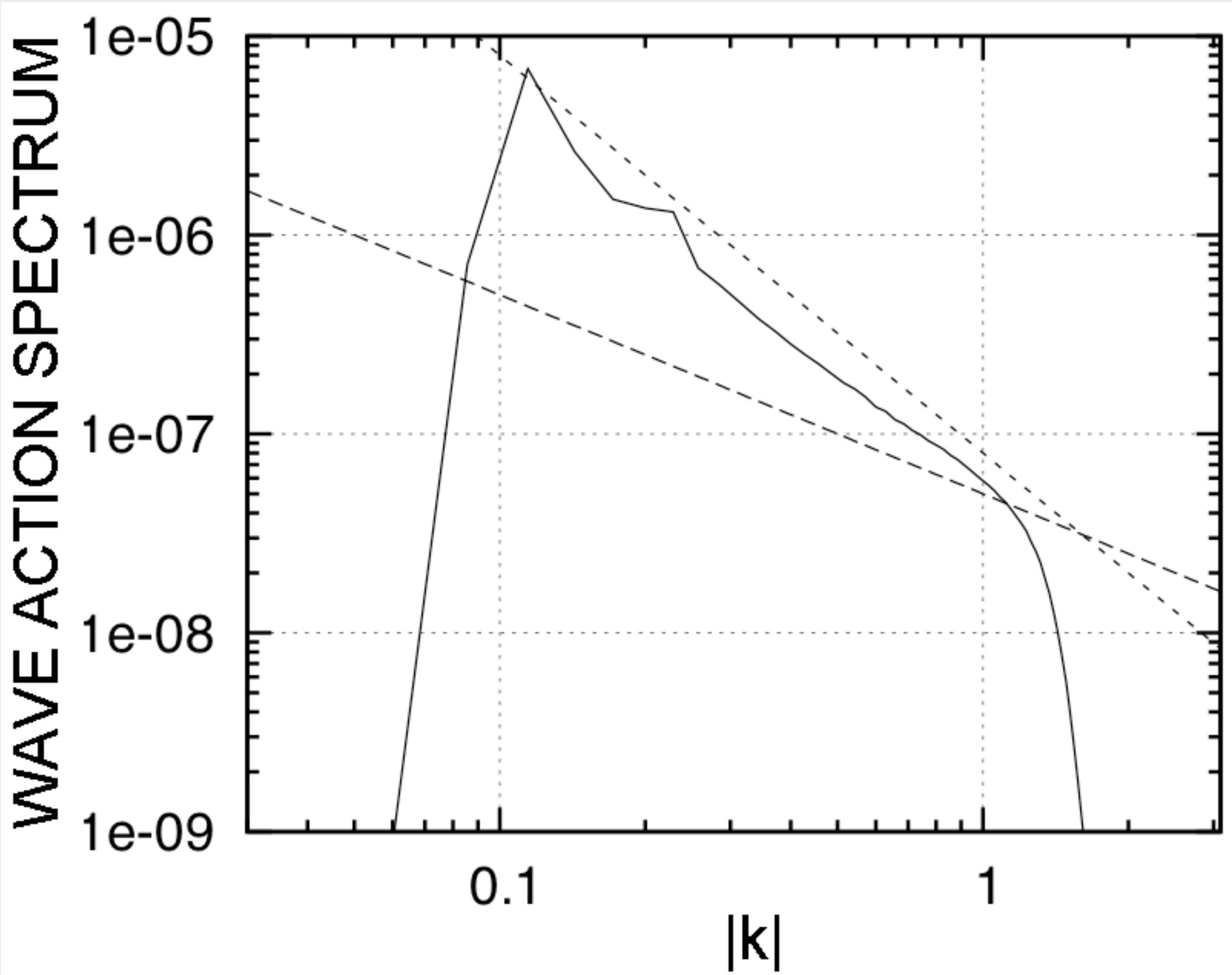



ω
6
4
2
0
-2
-4
-6
0
0.5
1
1.5
2
2.5
3
1
0.8
0.6
0.4
0.2
0
k

DIFFERENT FORCING AMPLITUDES

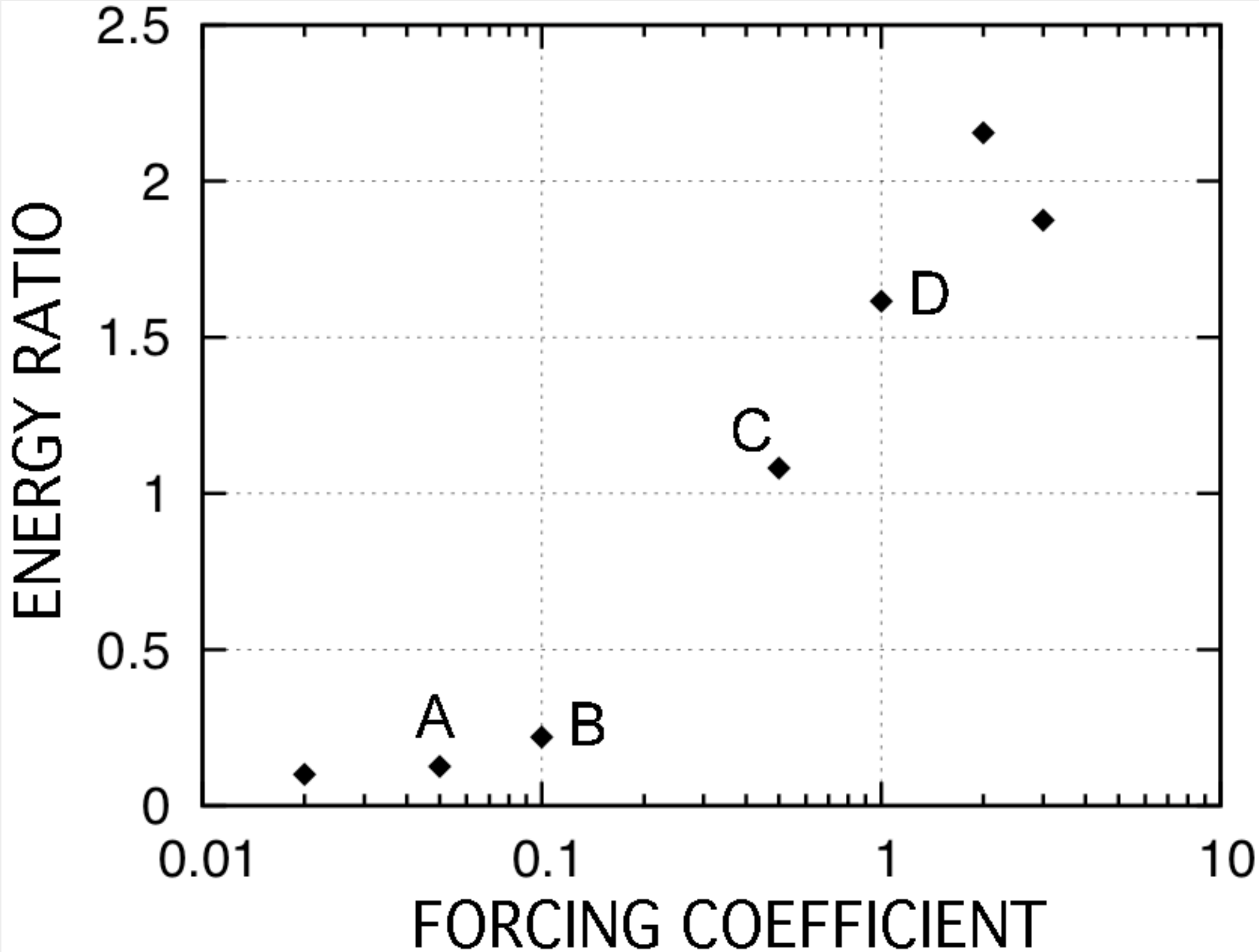


CASE A

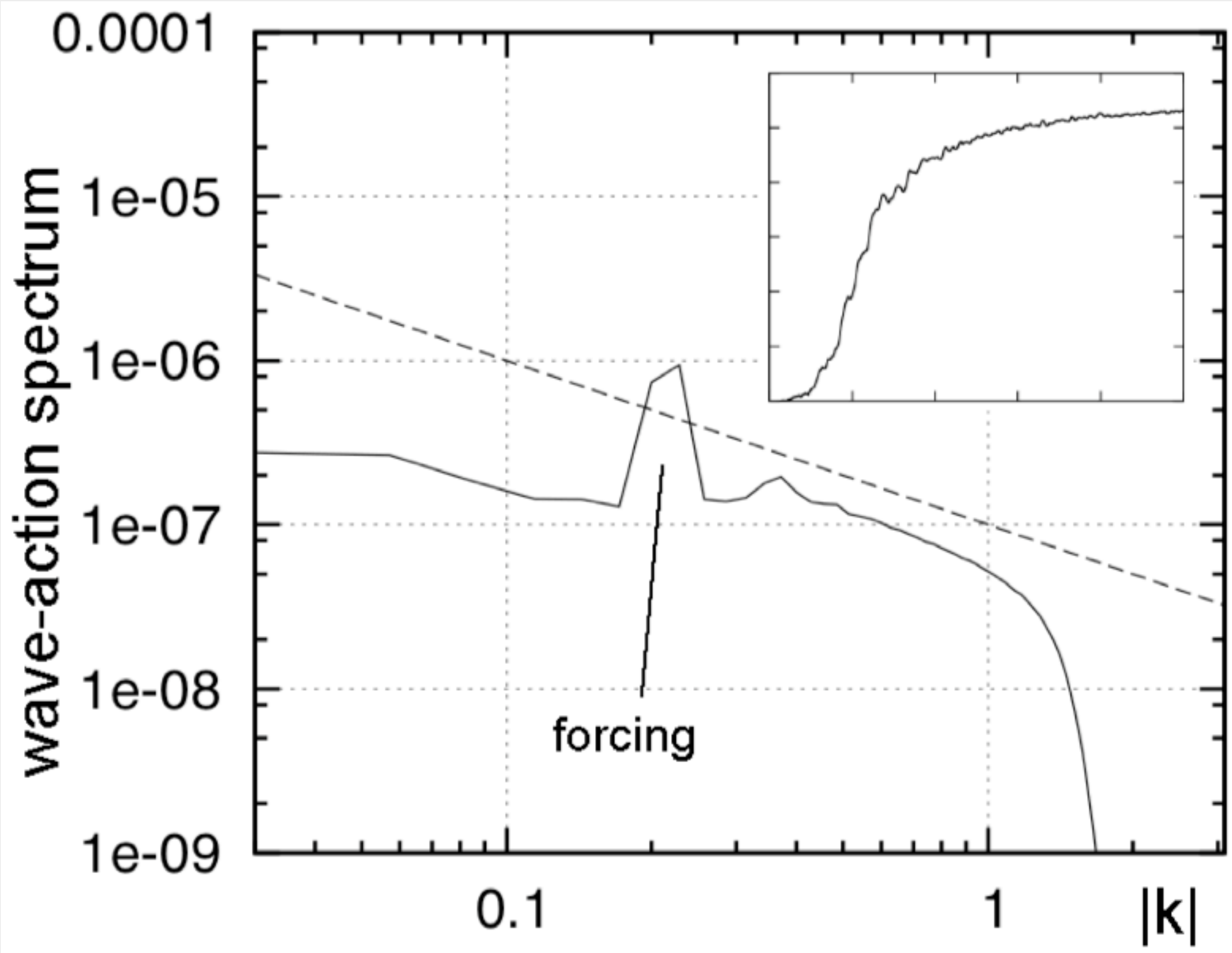


$$H = \frac{1}{2}\int |\nabla \psi|^2 d\mathbf{x} + \frac{1}{4}\int |\psi|^4 d\mathbf{x} = H_{LIN} + H_{NL}$$

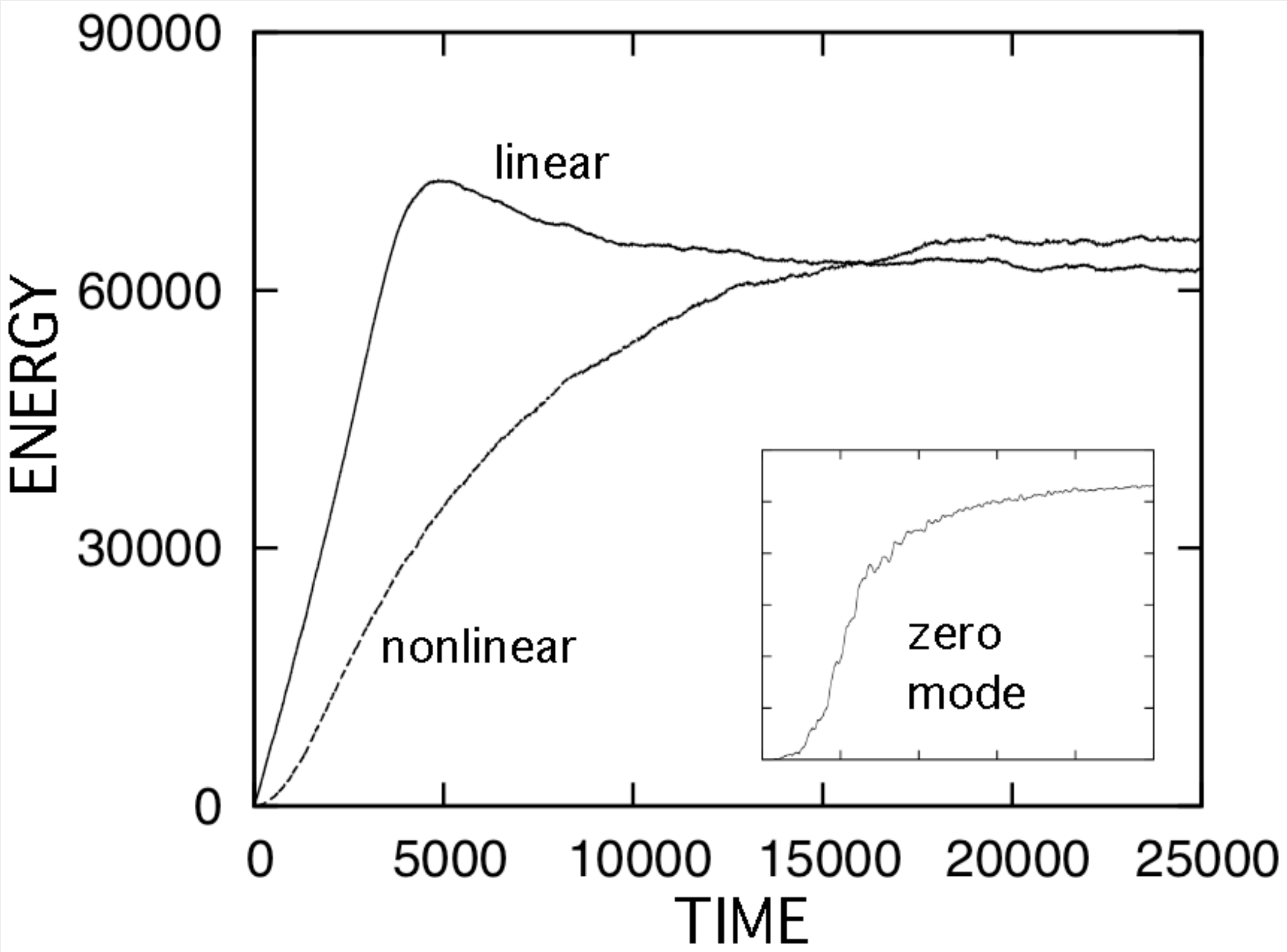
Energy *ratio*$=H_{NL}/H_{LIN}$

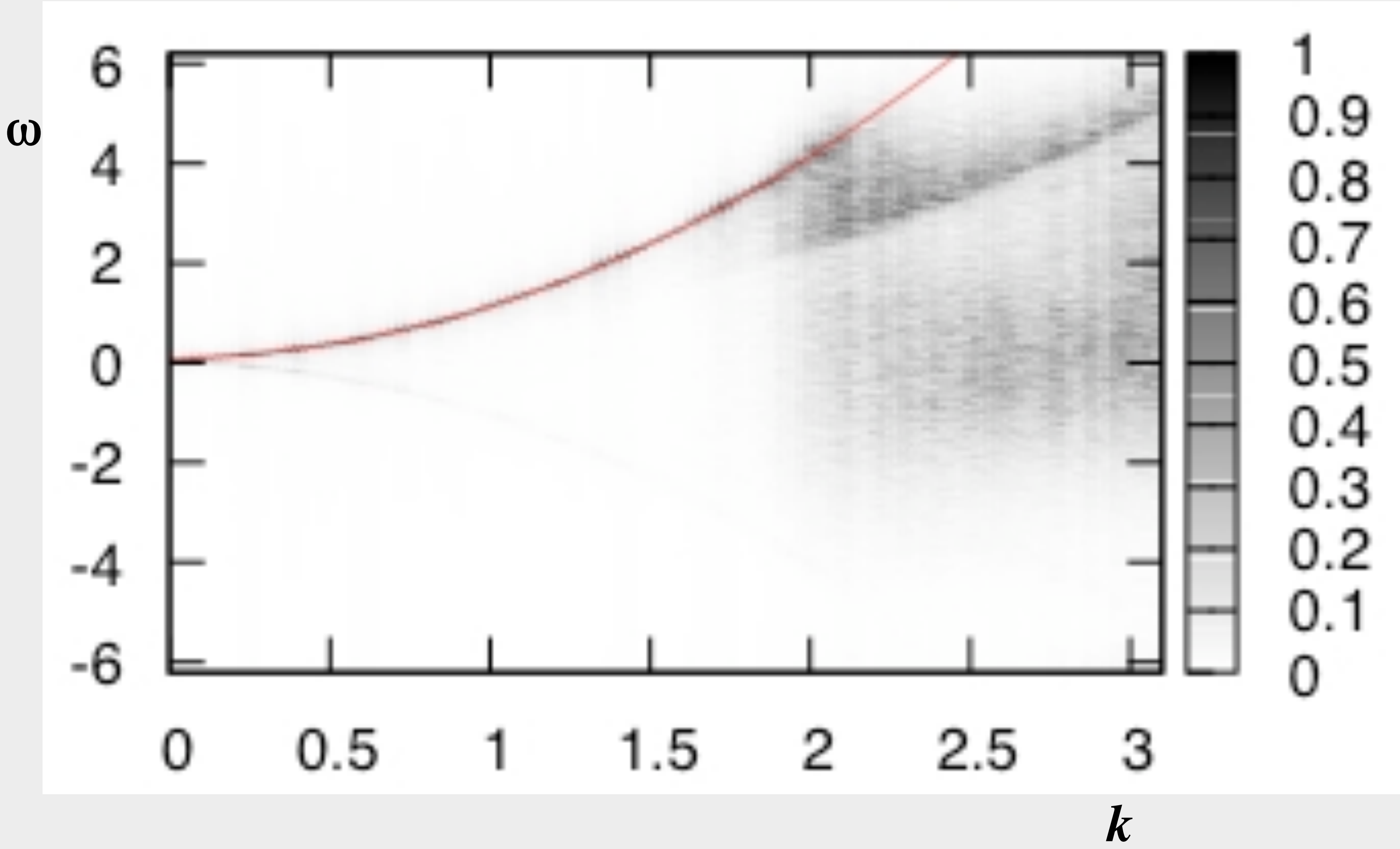


WITH FRICTION



$$H = \frac{1}{2}\int |\nabla \psi|^2 d\mathbf{x} + \frac{1}{4}\int |\psi|^4 d\mathbf{x} = H_{LIN} + H_{NL}$$



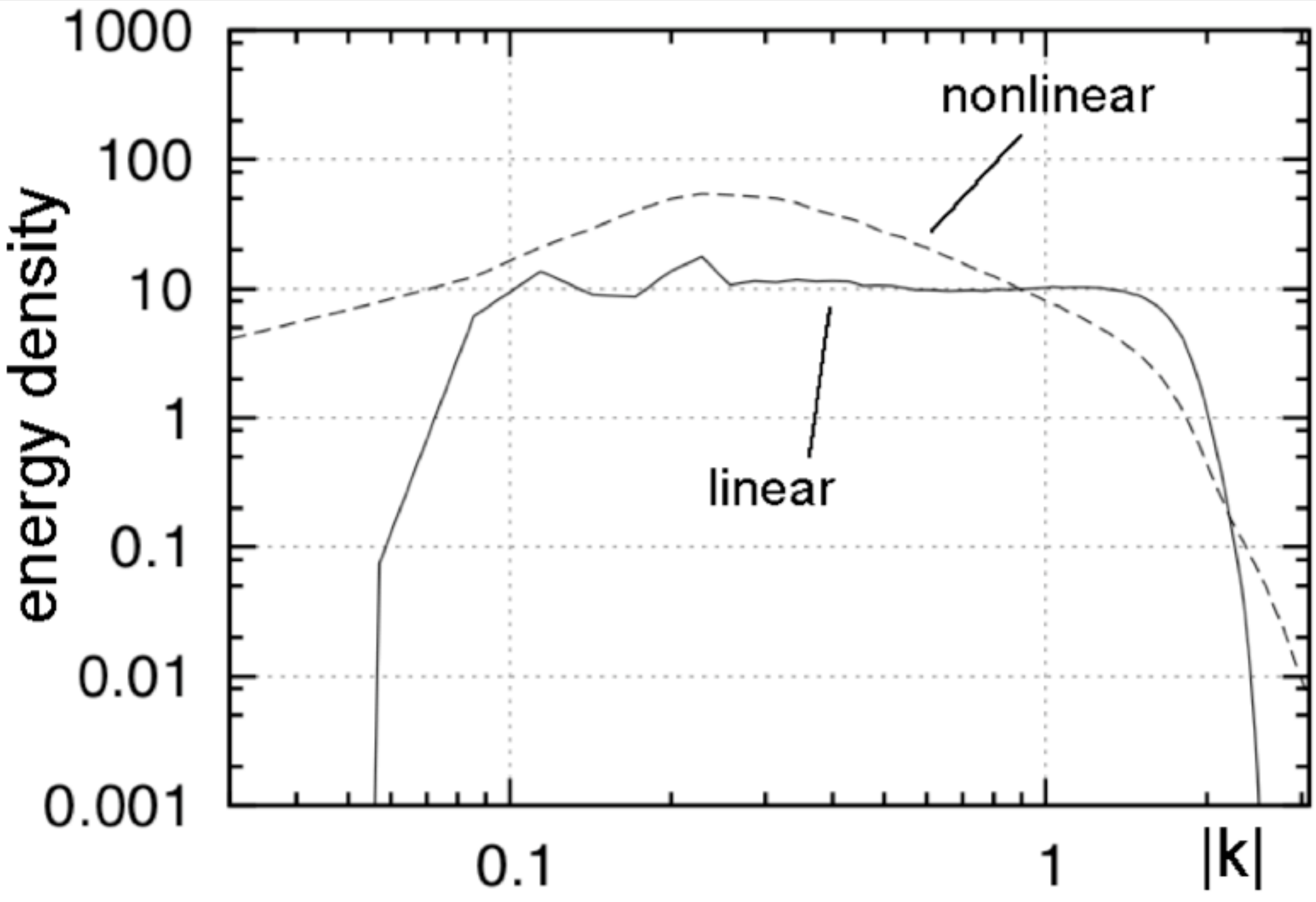


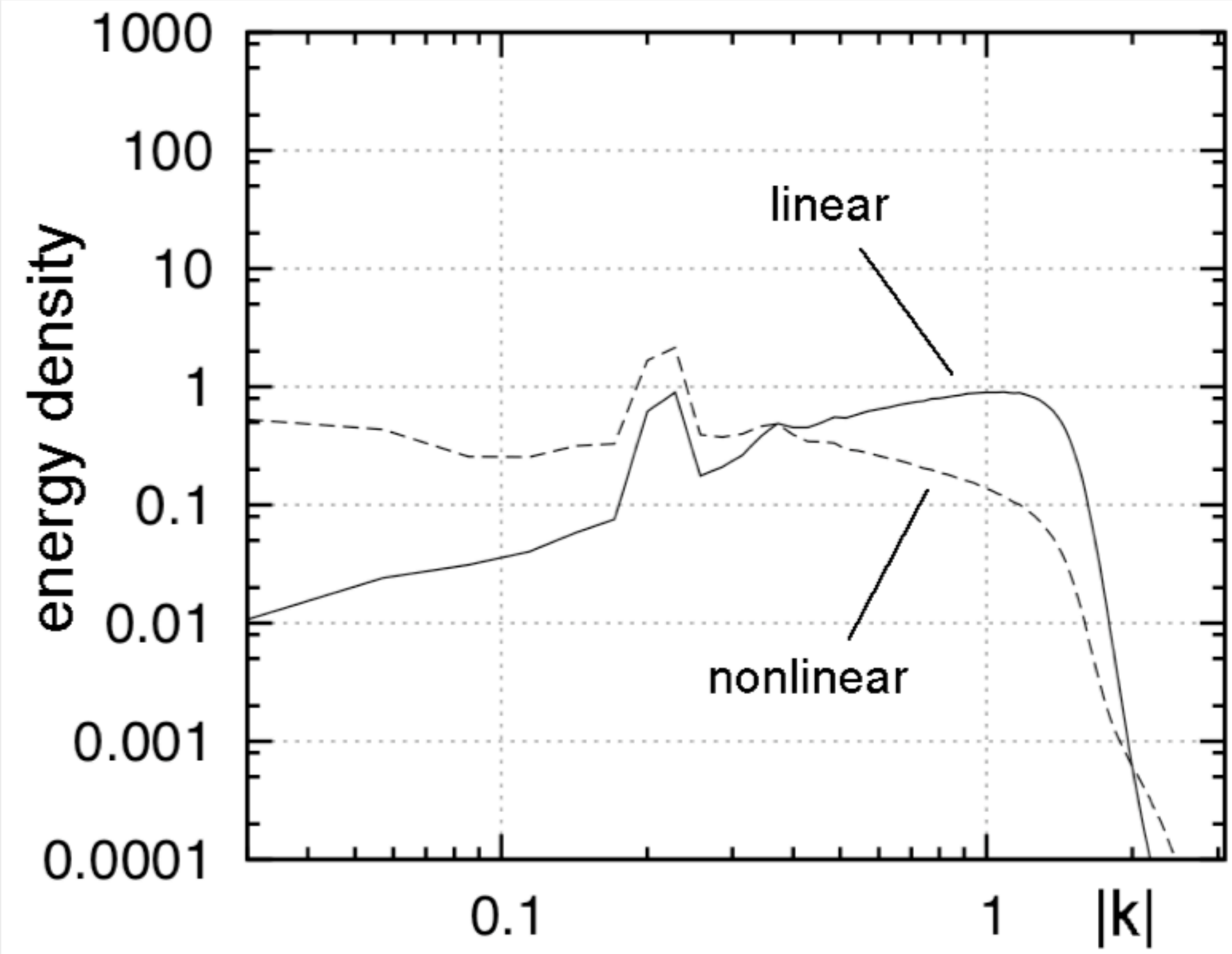
ω
6
4
2
0
-2
-4
-6
0
0.5
1
1.5
2
2.5
3
k
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

SCALE BY SCALE CRITICAL BALANCE

$$k^2\psi_k \sim |\psi_k|^2 \psi_k k^6$$

prediction: $n_k^{(1D)} \sim k^{-2}$


1000
100
10
1
0.1
0.01
0.001
energy density
nonlinear
linear
0.1
1
|k|


1000
100
10
1
0.1
0.01
0.001
0.0001
energy density
linear
nonlinear
0.1
1
|k|

SUMMARY

- Numerical simulations of the GPE in direct cascade

- Three wave interactions and observation of the Bogoliubov dispersion relation

- The dissipation mechanism at low wave number seems to be relevant for direct cascade

- Two class of power law solutions in the GPE

- Critical balance