Chaotic Waterwheel

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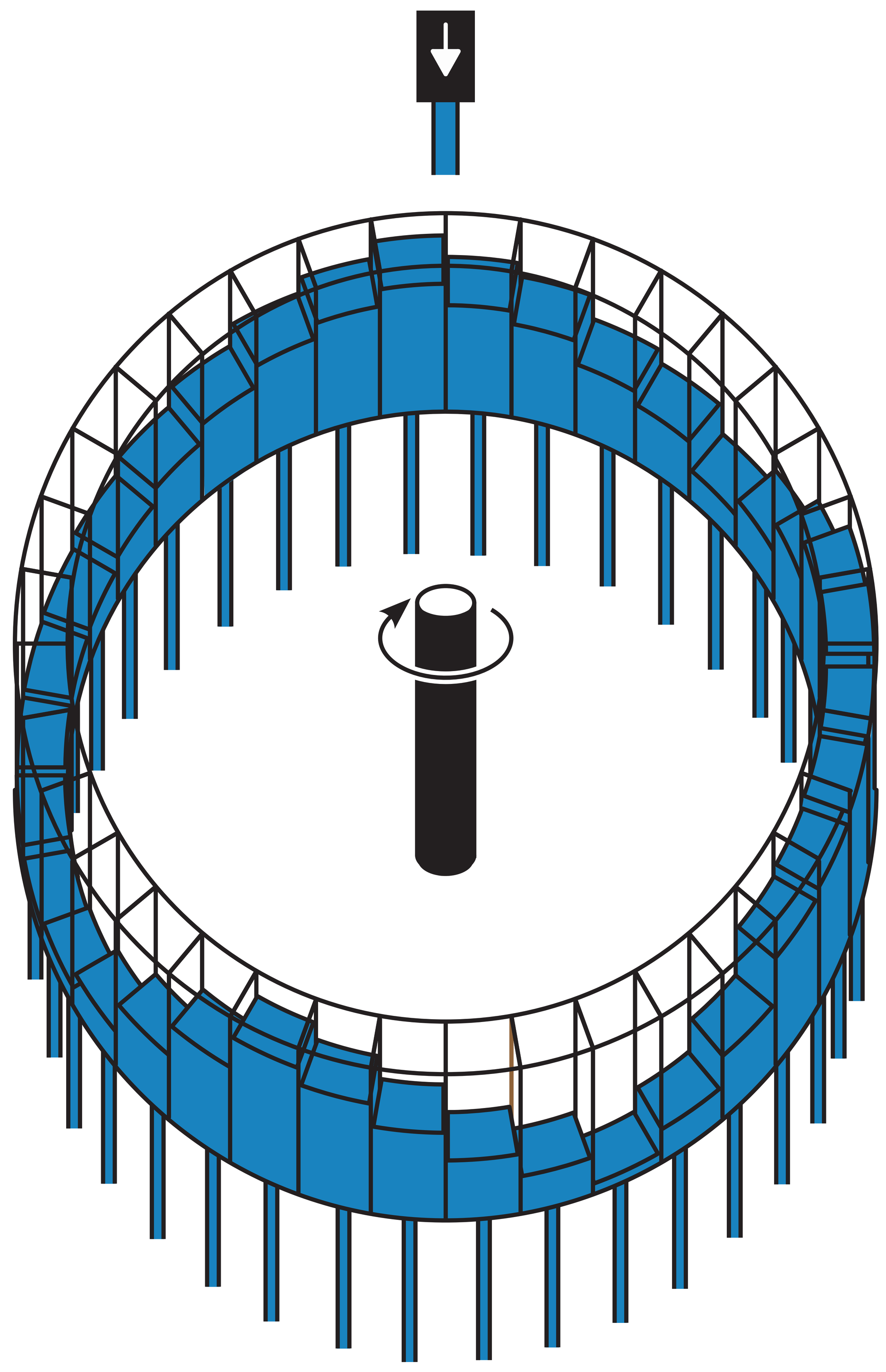
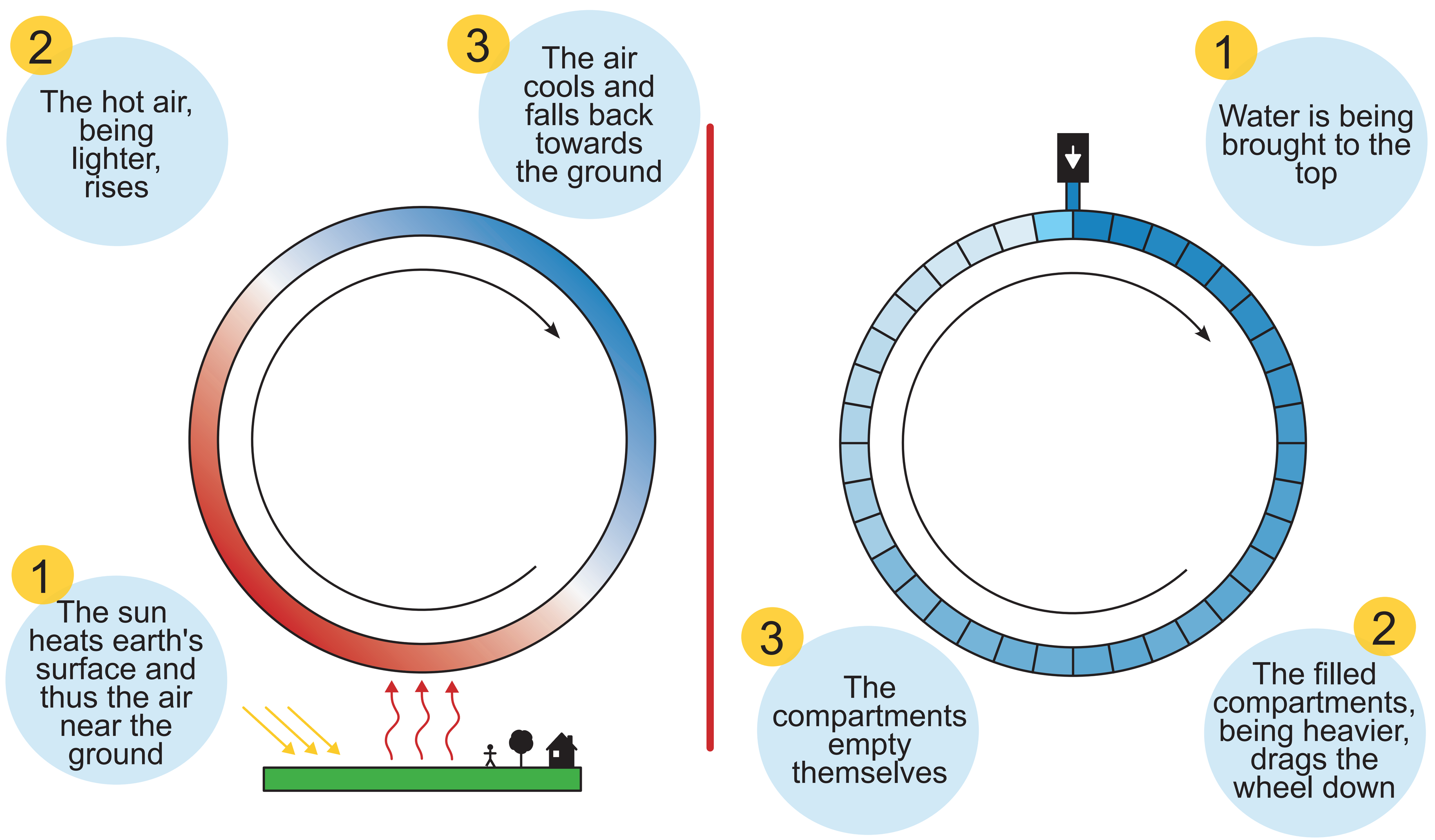
The device

Water is injected at the top of the wheel with a flow rate Q. This creates and maintains an unstable situation where the higher compartments, being heavier, fall down and drag the wheel with them.
The inertia of the wheel tends to make it keep the same rotation rate, while viscous friction ν and the compartments being slowly drained with an emptying rate K tend to slow it down.
As a result, the wheel can keep spinning in the same direction or change its direction of rotation depending on the value of the parameters and the dynamical state of the system.



This kind of waterwheel was introduced in the 1970's because they provide a experimental realization of Lorenz equations as well as a visual representation of the otherwise abstract chaos theory.
Indeed, the behaviour of the wheel, a simple mechanical system, is obviously deterministic. Depending on the values of the parameters, it can behave in a predictible way, for example being at a stop, spinning continuously, or changing of direction in a pendulum-like motion. However in some cases, it can exhibit sensitivity to initial conditions and chaotic dynamics, being as impredictible as a stochastic system.

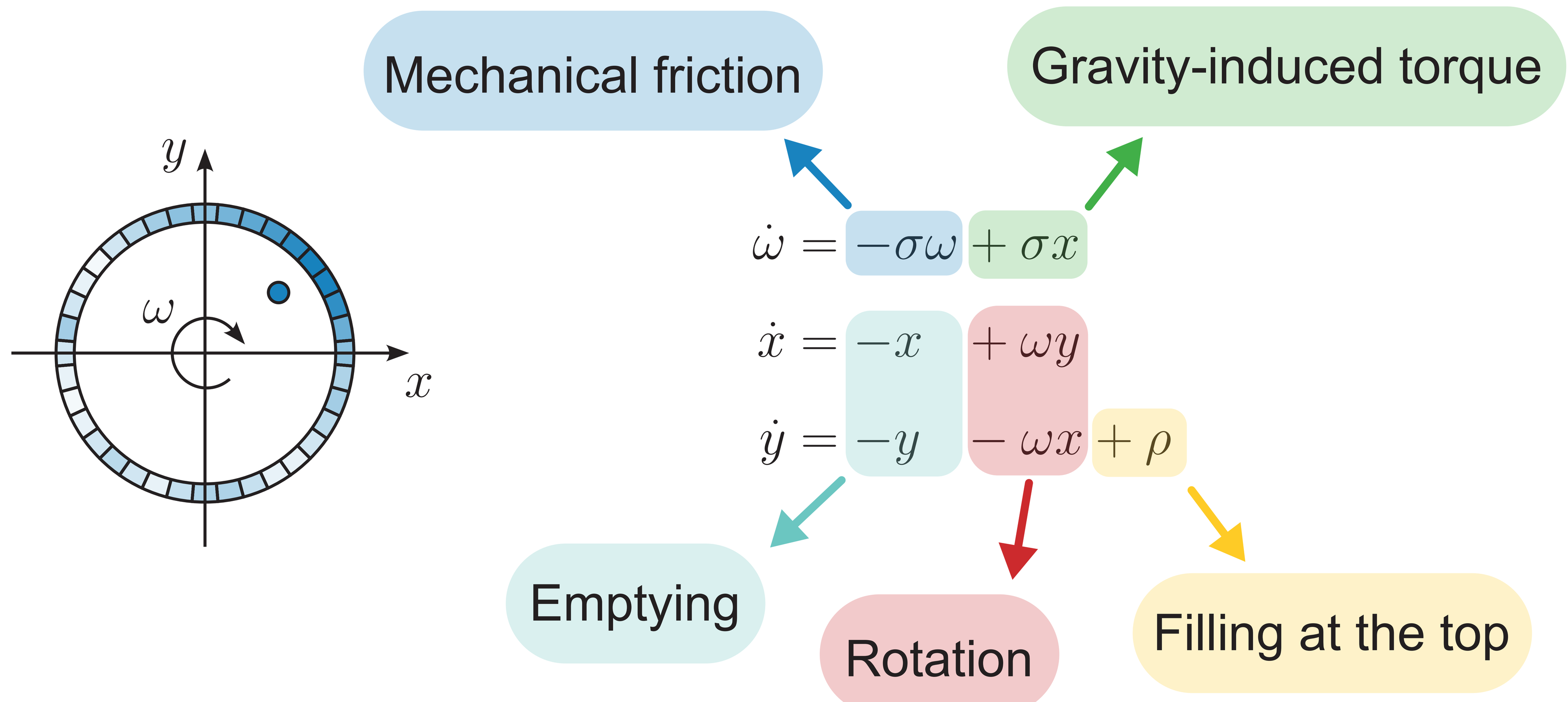
The waterwheel as a weather analog



The waterwheel can be seen as a mechanical analog for a single convection roll in the atmosphere. Its sensitivity to initial conditions can be related to the fact that weather systems are notoriously hard to predict.

The equations

Even though the mass repartition of the wheel is continuous, it is possible to reduce the dynamics of the system to a three-dimensional system of nonlinear differential equations, formally equivalent to the Lorenz system. The variables are the rotation speed of the wheel and the coordinates of the center of mass of the wheel. Each of the mathematical terms have a simple interpretation.



$$\dot{\omega} = -\sigma\omega + \sigma x$$
$$\dot{x} = -x + \omega y$$
$$\dot{y} = -y - \omega x + \rho$$

Despite there being more than seven parameters that we can vary, the dynamical behaviour of the wheel only depends on two dimensionless numbers, named by analogy with fluid dynamics.

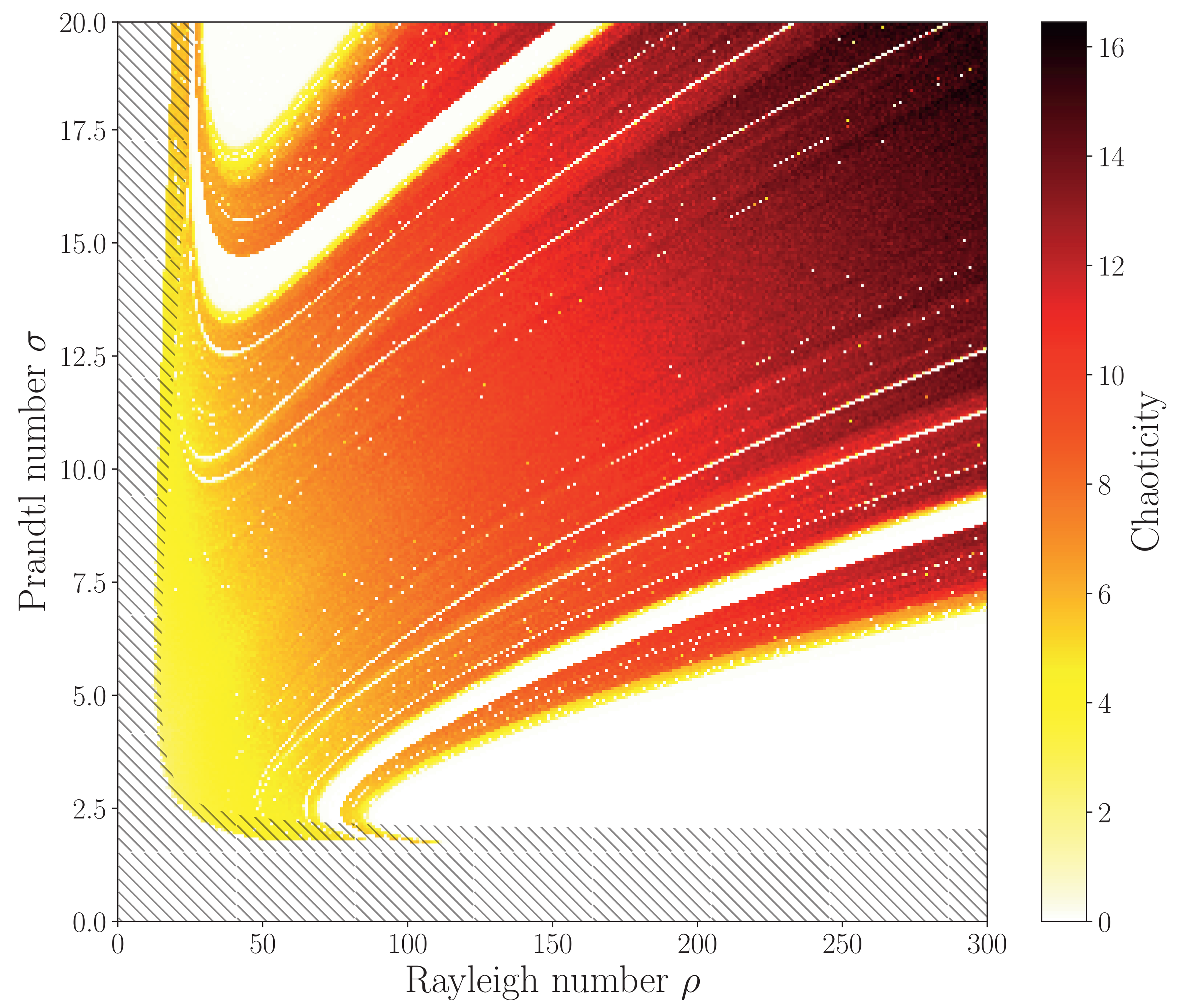
The **Rayleigh number** compares the gain in angular momentum due to the torque exerted by the heavier portions of the wheel to the loss due to mechanical friction and the emptying of the compartments.

$$\rho = \frac{Q\,R\,g}{K^2\,\nu}$$

The **Prandtl number** compares two kind of energy losses: mechanical friction and emptying of the water contained in the wheel. It is the ratio of the typical stopping times due to each of these effects.

$$\sigma = \frac{\nu}{M R^2 K}$$

Phase space structure



It is possible to numerically integrate the equations ruling the behaviour of the system and to plot the chaoticity, i.e. the impredictability, of the result.
Here we see periodic region alterned with chaotic regions, in a fractal fashion. In theory, periodic orbits are dense in the phase space.
The shaded region is the portion of the phase space where the wheel spinning at a constant rate is linearly stable.

See online abstract for references

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